Robertson's conjecture in topological combinatorics

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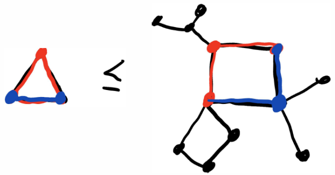

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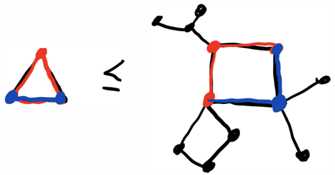

- One of the great early triumphs of topological graph theory is the theorem of Kuratowski, which states that **A graph can be embedded into the plane if and only if neither K_5 nor $K_{3,3}$ appear as a topological minor**. In other words, containment in the topological minor closed family of planar graphs is decided by a **finite** list of forbidden minors.

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- It turns out the answer is **no** as the following family illustrates.

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- Very recently, Liu and Thomas [LT] proved a conjecture of Robertson that made this observation precise:

Robertson's Conjecture

Let $\mathcal{G}_d$ denote the class of graphs which do not contain a Robertson chain of length d or higher as a topological minor. Then containment in any topological minor closed family within $\mathcal{G}_d$ is determined by a finite list of forbidden minors.

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Definition: Configuration Space

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- These spaces arise naturally in topological robotics and motion planning questions.
- What will be relevant to us is the following list of theorems, each of which points to some kind of underlying uniformity of these spaces across all graphs.

Uniform boundedness of Torsion [KP]

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Universality of planar generators for H_2 [AK]

There exists a finite set of graphs, depending only on n, such that for any planar graph G, $H_2(\mathcal{F}_n(G))$ is generated by push forwards of classes along embeddings of the members of this finite list into G.

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- Fix for now an integer d, and suppose we are working only with graphs not containing Robertson chains of length longer than d (e.g. in $\mathcal{G}_d$). We observe that if $G \leq G'$, then for any $n \geq 1$ there is a natural continuous map $\mathcal{F}_n(G) \to \mathcal{F}_n(G')$.

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- These continuous maps at the level of topological spaces will induce homomorphisms for each $i \geq 0$ at the level of homology groups $H_i(\mathcal{F}_n(G)) \to H_i(\mathcal{F}_n(G'))$.

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- At this point, we find ourselves in the realm of "categorification," where combinatorial objects become vector spaces, and maps become homomorphisms. Our dream is now that this categorification is Robust enough that it carries the most important bits of combinatorics (e.g. Robertson's conjecture) into the realm of algebra. We make this dream precise with the following.

Definition: $\mathcal{G}_d$-modules and finite generation

For any integer $d \geq 1$, a $\mathcal{G}_d$-module is a collection of abelian groups $\{V(G)\}$, one for each graph in $\mathcal{G}_d$, such that whenever $G \leq G'$ there is a natural homomorphism $V(G) \to V(G')$. We say V is **finitely generated** whenever there is a finite list of graphs in $\mathcal{G}_d$, $\{G_1, \ldots, G_r\}$ such that for any graph G, $V(G)$ is spanned by classes coming from the groups $V(G_i)$.

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Corollary: Finite generation of graph configuration spaces

For any integers $d, i, n \geq 1$, the $\mathcal{G}_d$-module $H_i(\mathcal{F}_n(\bullet))$ is finitely generated.

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