Cluster realization of Weyl groups and applications to representation theory

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with

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§0 Introduction

- Cluster algebra [Fomin-Zelevinsky 00]

$I = \{1, 2, \ldots, N\}$: a finite set, $t = (\varepsilon, \boldsymbol{A}, \boldsymbol{X})$: a seed

$\varepsilon = (\varepsilon_{ij})_{i,j \in I}$: a skew symmetrizable matrix

$\exists d = \mathrm{diag}(d_i)_{i \in I}$; $d_i \in \mathbb{Z}_{>0}$ s.t. εd is skew sym, $\gcd\{d_i\}_i = 1$

$\boldsymbol{A} = (A_i)_{i \in I}$: cluster variables ($A$-var)

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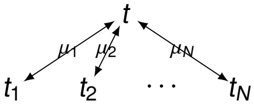

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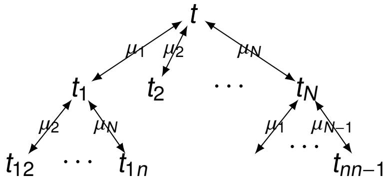

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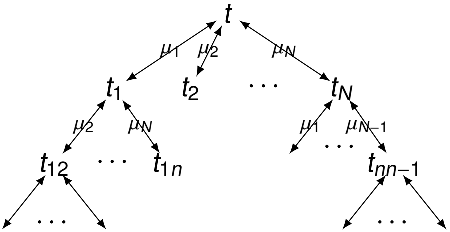

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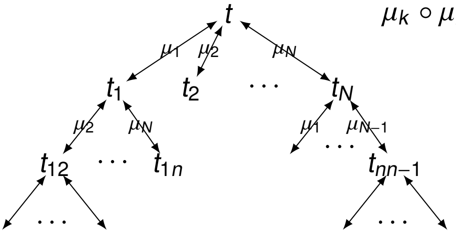

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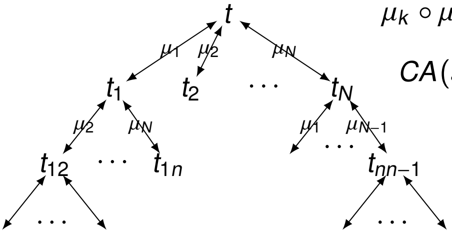

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$CA(\varepsilon, \boldsymbol{A}) := \mathbb{Z}[A\text{-var. in all seeds}]$

$$\mu_k(\varepsilon, \boldsymbol{A}, \boldsymbol{X}) = (\widetilde{\varepsilon}, \widetilde{\boldsymbol{A}}, \widetilde{\boldsymbol{X}});\ k \in I:$$

$$\widetilde{X}_i = \begin{cases} {X_k}^{-1} & i = k \\ X_i\,(1 + X_k^{-\mathrm{sgn}(\varepsilon_{ik})})^{-\varepsilon_{ik}} & i \neq k \end{cases} \qquad \widetilde{A}_i = \begin{cases} \dfrac{\prod_{j:\varepsilon_{kj}>0} {A_j}^{\varepsilon_{kj}} + \prod_{j:\varepsilon_{kj}<0} {A_j}^{-\varepsilon_{kj}}}{A_k} & i = k \\ A_i \quad \text{for } i \neq k \end{cases}$$

$$\widetilde{\varepsilon}_{ij} = \begin{cases} -\varepsilon_{ij} & i = k \text{ or } j = k \\ \varepsilon_{ij} + \dfrac{|\varepsilon_{ik}|\varepsilon_{kj} + \varepsilon_{ik}|\varepsilon_{kj}|}{2} & \text{ow} \end{cases}$$

Remarks

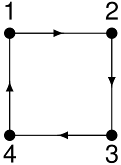
· $\varepsilon_{ij} = 0 \Longleftrightarrow \mu_i \circ \mu_j = \mu_j \circ \mu_i$

· $(\varepsilon, d) \overset{1:1}{\longleftrightarrow} Q = (\sigma, d)$: weighted quiver (w/o 1-loop, 2-cycle)

$$\mathrm{wt}(i) = d_i,\ \sigma_{ij} = \varepsilon_{ij}\frac{\mathrm{gcd}(d_i, d_j)}{d_i}$$

$$\sigma_{ij} = \#\{i \to j\} - \#\{j \to i\}$$

(Ex)



$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix}$$

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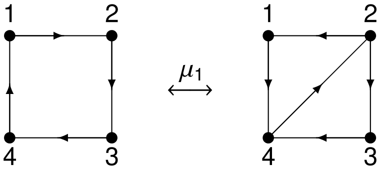

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} \qquad \begin{pmatrix} (A_4 + A_2)/A_1 := A_1' \\ A_2 \\ A_3 \\ A_4 \end{pmatrix}$$

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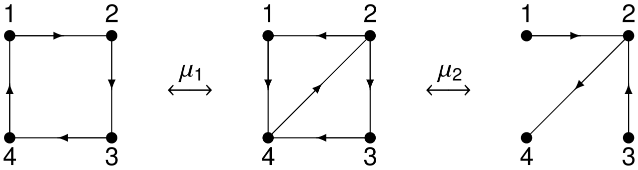

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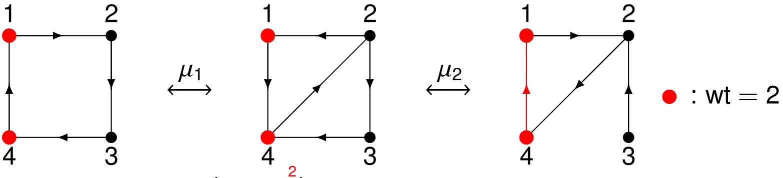
(Ex)



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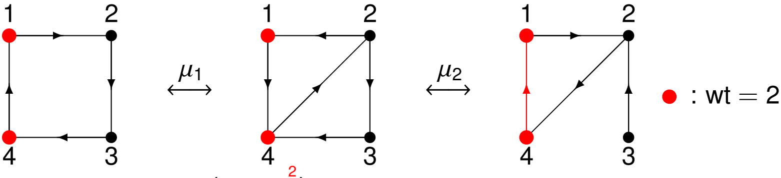

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<u>Idea</u>

a quiver Q and a sequence of mutations $\mu_{i_1}\mu_{i_2}\cdots\mu_{i_p}$ preserving Q

$\rightsquigarrow$ a rational map on $\mathbb{C}(\boldsymbol{A})$ and $\mathbb{C}(\boldsymbol{X})$

Cluster realization of $W(\mathfrak{g})$

$Qm(\mathfrak{g})$ $Qm'(\mathfrak{g})$

- geometric R
  $\mathfrak{g} = A_\ell$ [I-Lam-Pylyavskyy 19]
- q-Painlevé eq.
  $Q_2(A_2^{(1)})$
  [Bershtein-Gavrylenko-Marshakov 18]
  [Okubo-Suzuki 22]
  [Masuda-Okubo-Tsuda 21]

- Higher Teichmüller theory
  [Fock-Goncharov 06]
  [Goncharov-Shen 18, 19]
  [I-Ishibashi-Oya 21]
- Positive rep of $U_q(\mathfrak{g})$
  [Schrader-Shapiro 19]
  [Ip 18]

- q-character of $U_q(\hat{\mathfrak{g}})$
  [Hernandez-Leclerc 16]
  [I 20]
  [I-Yamazaki 22]
  ⋮
  [Frenkel-Hernandez 22]

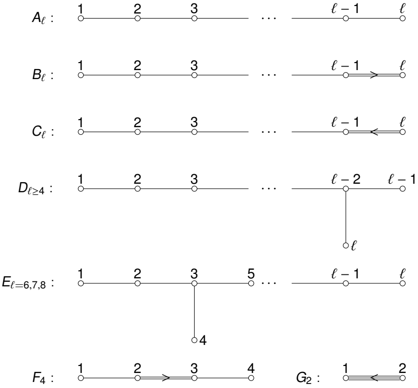
§1 Cluster realization of Weyl group

- Lie algebras

$\mathfrak{g}$: a fin dim simple Lie alg of rank ℓ, $S := \{1, 2, \ldots, \ell\}$

$C = (C_{ij})_{i,j \in S}$: the Cartan mat; $C_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$

$D_i = \frac{(\alpha_i,\alpha_i)}{2}$, $D := \min(D_i)_{i \in S}$, $D' := \max(D_i)_{i \in S}$



- Quivers $Q_m(\mathfrak{g})$, $Q'_m(\mathfrak{g})$

Def (a quiver $Q_m(\mathfrak{g})$) [I-Ishibashi-Oya 21]

the Dynkin quiver for $\mathfrak{g}$:

$\overset{i}{\bullet}\longleftarrow\overset{j}{\bullet}$; $C_{ij} < 0$ and $i < j$

$m > 1$ copies on a cylinder

$\rightsquigarrow Q_m(\mathfrak{g})$; $I = \{v^i_n;\ i \in S,\ n \in \mathbb{Z}/m\mathbb{Z}\}$

$\mathrm{wt}(v^i_n) = \frac{D_i}{D}$ $\left(= \frac{(\alpha_i,\alpha_i)}{\min_j(\alpha_j,\alpha_j)}\right)$

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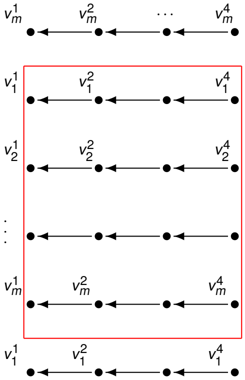

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(Ex) $\mathfrak{g} = A_4$; $\mathrm{wt}(v_n^i) = 1\ (i \in S)$

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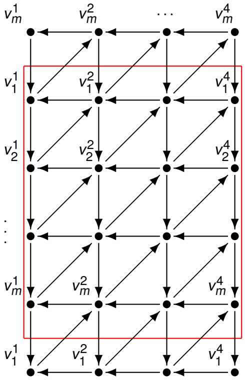

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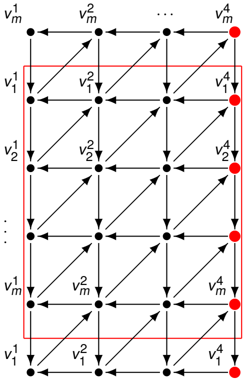

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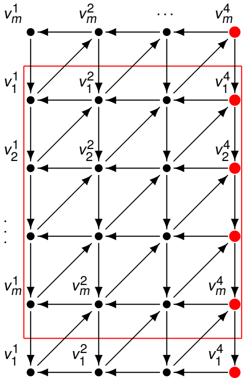
$$\mathfrak{g} = C_4;\ \mathrm{wt}(v_n^i) = \begin{cases} 1 & i = 1,2,3 \\ 2 & i = 4 \end{cases}$$

- Quivers $Q_m(\mathfrak{g})$, $Q'_m(\mathfrak{g})$



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For $i \in S$,

P_i: directed closed path $v_1^i \to v_2^i \to \cdots \to v_m^i \to v_1^i$

Def (quiver $Q'_m(\mathfrak{g})$) [I 21] (Cf. [Hernandez-Leclerc16])

$\mathfrak{g}$ simply-laced: $Q'_m(\mathfrak{g}) = Q_m(\mathfrak{g})$

$\mathfrak{g}$ nonsimply-laced ($\exists i \in S, D_i \neq 1$):

$Q'_m(\mathfrak{g})$; $I' = \{v^i_n;\ i \in S, n \in D\mathbb{Z}/D'm\mathbb{Z}\}$, $\mathrm{wt}(v^i_n) = 1$

$D_i = \frac{(\alpha_i,\alpha_i)}{2}, D = \min(D_i)_{i\in S},\ D' = \max(D_i)_{i\in S}$

For $i \in S$, $\gamma_i = 1, 2, \ldots, \frac{D_i}{D}$,

P_{i,γ_i}: directed closed path in $Q'_m(\mathfrak{g})$

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For $i \in S$, $\gamma_i = 1, 2, \dots, \frac{D_i}{D}$,

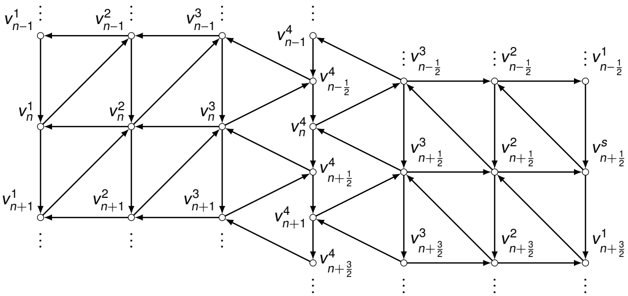
P_{i,γ_i}: directed closed path in $Q'_m(\mathfrak{g})$

$Q'_m(B_4)$:

$(D_i)_i = (1, 1, 1, \frac{1}{2})$

$P_{i,1}, P_{i,2}$ $(i = 1, 2, 3)$

$P_{4,1}$



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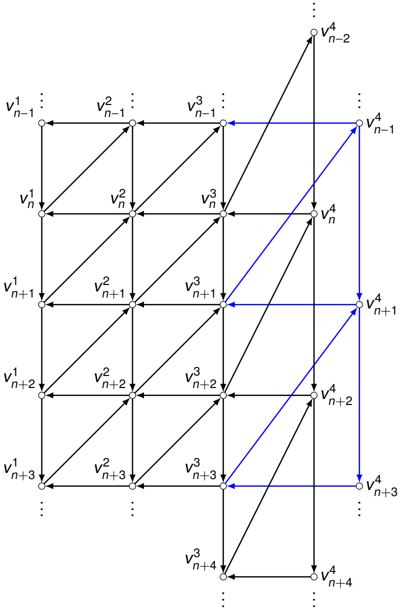

• Realization of Weyl group

Def [Bucher 14]

a directed closed path P; $v_1 \to v_2 \to \cdots \to v_p \to v_1$,

$$R(P; v_1) := \mu_1 \cdots \mu_{p-2} \circ (v_{p-1}, v_p) \circ \mu_p \mu_{p-1} \circ \mu_{p-2} \cdots \mu_1$$

$$f_X(P; v_n) := 1 + \sum_{k=0}^{p-2} X_{v_n} X_{v_{n-1}} \cdots X_{v_{n-k}}; \ n \in \mathbb{Z}/p\mathbb{Z}$$

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$$\rightsquigarrow R(P; v_1)(P, \mathbf{A}, \mathbf{X}) = (P, \mathbf{A}', \mathbf{X}')$$

$$\begin{cases} A'_{v_n} = A_{v_n} \displaystyle\sum_{n \in \mathbb{Z}/p\mathbb{Z}} \frac{1}{A_{v_k} A_{v_{k+1}}} \\ X'_{v_n} = \dfrac{f_X(P; v_n)}{X_{v_{n-1}} f_X(P; v_{n-2})} \end{cases}$$

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Consider

$R(P_i; v_n^i)$, $f_X(P_i, v_n^i)$ for $Q_m(\mathfrak{g})$; $i \in S$,

$R(P_{i,\gamma_i}; v_n^i)$, $f_X(P_i, v_n^i)$ for $Q'_m(\mathfrak{g})$; $i \in S$, $v_n^i \in P_{i,\gamma_i}$.

Thm 1 [I-Lam-Pylyavskyy 19] [IIO 21]

(i) $R(P_i, v_n^i)$ preserves $Q_m(\mathfrak{g})$, and the action on the seed does not depend on v_n^i.

$\rightsquigarrow R(i) := R(P_i, v_n^i)$.

(ii) $R(i)$ ($i \in S$) generate the Weyl group $W(\mathfrak{g})$ action on the rational functional fields $\mathbb{C}(\boldsymbol{X})$, $\mathbb{C}(\boldsymbol{A})$, i.e., $(R(i)R(j))^{m_{ij}} = 1$.

Here $m_{ii} = 1$, and for $i \neq j$

$C_{ij}C_{ji}$	0	1	2	3
m_{ij}	2	3	4	6

Thm 1 [I-Lam-Pylyavskyy 19] [IIO 21]

(i) $R(P_i, v_n^i)$ preserves $Q_m(\mathfrak{g})$, and the action on the seed does not depend on v_n^i.

$\rightsquigarrow R(i) := R(P_i, v_n^i)$.

(ii) $R(i)$ $(i \in S)$ generate the Weyl group $W(\mathfrak{g})$ action on the rational functional fields $\mathbb{C}(\boldsymbol{X})$, $\mathbb{C}(\boldsymbol{A})$, i.e., $(R(i)R(j))^{m_{ij}} = 1$.

Here $m_{ii} = 1$, and for $i \neq j$

$C_{ij}C_{ji}$	0	1	2	3
m_{ij}	2	3	4	6

Thm 2 [I 21]

(i) $R(P_{i,\gamma_i}, v_n^i)$ preserves $Q'_m(\mathfrak{g})$, and the action on the seed does not depend on v_n^i. $R(P_{i,\gamma_i}, v_n^i)(\gamma_i = 1, \cdots, \frac{D_i}{D})$ commute.

$\rightsquigarrow R_i := \prod_{\gamma_i = 1, \cdots, \frac{D_i}{D}} R(P_{i,\gamma_i}, v_n^i)$.

(ii) R_i $(i \in S)$ generate the $W(\mathfrak{g})$-action on $\mathbb{C}(\boldsymbol{X})$, $\mathbb{C}(\boldsymbol{A})$, i.e., the R_i satisfy $(R_iR_j)^{m_{ij}} = 1$.

$f_X(P; v_n) := 1 + \sum_{k=0}^{p-2} X_{v_n} X_{v_{n-1}} \cdots X_{v_{n-k}} = 1 + X_{v_n} + X_{v_n} X_{v_{n-1}} + \cdots + X_{v_n} X_{v_{n-1}} \cdots X_{v_{n+2}}$

For $v_n^i \in P_i, P_{i,\gamma_i}$, write $f_X(i,n)$ for $f_X(P_i, v_n^i)$ or $f_X(P_{i,\gamma_i}; v_n^i)$

(Ex) Action on X-var : $\mathfrak{g} = A_3$ (C_3)

$$(D_i)_{i \in S} = (1, 1, 1(2)), \quad C(A_3(C_3)) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1(-2) \\ 0 & -1 & 2 \end{pmatrix},$$

$Q_m(\mathfrak{g})$: $\mathfrak{g} = A_3, (C_3)$

$$R(2)^*(X_n^1, X_n^2, X_n^3) = \left(X_n^1 \frac{X_{n-1}^2 f_X(2, n-2)}{f_X(n-1)}, \frac{f_X(2,n)}{X_{n-1}^2 f_X(2, n-2)}, \underline{X_n^3 \left(\frac{X_n^2 f_X(2, n-1)}{f_X(2,n)} \right)^{1(2)}} \right)$$

$$R(3)^*(X_n^1, X_n^2, X_n^3) = \left(X_n^1, X_n^2 \frac{X_{n-1}^3 f_X(3, n-2)}{f_X(3, n-1)}, \underline{\frac{f_X(3,n)}{X_{n-1}^3 f_X(3, n-2)}} \right)$$

$f_X(P; v_n) := 1+\sum_{k=0}^{p-2} X_{v_n}X_{v_{n-1}}\cdots X_{v_{n-k}} = 1+X_{v_n}+X_{v_n}X_{v_{n-1}}+\cdots+X_{v_n}X_{v_{n-1}}\cdots X_{v_{n+2}}$

For $v_n^i \in P_i, P_{i,\gamma_i}$, write $f_X(i,n)$ for $f_X(P_i, v_n^i)$ or $f_X(P_{i,\gamma_i}; v_n^i)$

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$Q'_m(\mathfrak{g})$: $\mathfrak{g} = C_3$

$$R_2^*(X_n^1, X_n^2, X_n^3) = \left(X_n^1 \frac{X_{n-1}^2 f_X(2,n-2)}{f_X(n-1)}, \frac{f_X(2,n)}{X_{n-1}^2 f_X(2,n-2)}, X_n^3 \frac{X_n^2 X_{n+1}^2 f_X(2,n-1)}{f_X(2,n+1)} \right)$$

$$R_3^*(X_n^1, X_n^2, X_n^3) = \left(X_n^1, X_n^2 \frac{X_{n-1}^3 f_X(3,n-2)}{f_X(3,n-1)}, \frac{f_X(3,n)}{X_{n-2}^3 f_X(3,n-4)} \right)$$

Remark

Both for $Q_m(\mathfrak{g})$, $Q'_m(\mathfrak{g})$, the $W(\mathfrak{g})$-action on $\mathbb{C}(\mathbf{X})$ is an analogue of the Weyl group action on the root system.
For $Q_m(\mathfrak{g})$, the induced action on $\mathbb{X}_k := \prod_{v_n^k \in P_k} X_n^k$ is

$$R(i)^*(\mathbb{X}_k) = \mathbb{X}_k\, \mathbb{X}_i^{-C_{ik}}$$

which corresponds to the action on a simple root α_k:
$r_i\, \alpha_k = \alpha_k - C_{ik}\, \alpha_i$.
It is similar for $Q'_m(\mathfrak{g})$.

§2 Application of $Q'_m(\mathfrak{g})$: q-character of $U_q(\hat{\mathfrak{g}})$

- q-character and screening operators [Frenkel-Reshetikhin 90s]

$U_q(\hat{\mathfrak{g}})$: an affine quantum grp; $q \in \mathbb{C}^\times$ (not root of unity)

χ_q : Rep $U_q(\hat{\mathfrak{g}}) \to \mathbf{Y} := \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in S, a \in \mathbb{C}^\times}$: q-character

Thm [Frenkel-Reshetikhin 99] [Frenkel-Mukhin 01]

$$\mathrm{Im}\,\chi_q = \bigcap_{i \in S} \Big(\mathbb{Z}[Y_{j,a}^{\pm 1}]_{j \neq i, a \in \mathbb{C}^\times} \otimes \mathbb{Z}[Y_{i,b}(1 + A_{i,bq_i}^{-1})]_{b \in \mathbb{C}^\times} \Big); \quad q_i = q^{D_i}$$

$$A_{i,a} = Y_{i,aq_i} Y_{i,aq_i^{-1}} \prod_{j:C_{ji}=-1} Y_{j,a}^{-1} \prod_{j:C_{ji}=-2} Y_{j,aq_j}^{-1} Y_{j,aq_j^{-1}}^{-1} \prod_{j:C_{ji}=-3} Y_{j,aq_j^2}^{-1} Y_{j,a}^{-1} Y_{j,aq_j^{-2}}^{-1}.$$

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For $i \in S$,

$\mathcal{S}_i : \mathbf{Y} \to \mathbf{YS}_i := \oplus_{b \in \mathbb{C}^\times} \mathbf{Y} \otimes S_{i,b}$; $Y_{j,a} \mapsto \delta_{i,j} Y_{i,a} S_{i,a}$: screening operator

Consider the quotient space of $\mathbf{YS}_i$ with relation $S_{i,aq_i^2} = A_{i,aq_i} S_{i,a}$.

Thm [FR 99] [FM 01]

$$\bigcap_{i \in S} \operatorname{Ker}\mathcal{S}_i = \operatorname{Im}\chi_q; \;\; \operatorname{Ker}\mathcal{S}_i = \mathbb{Z}[Y_{j,a}^{\pm 1}]_{j \neq i, a \in \mathbb{C}^\times} \otimes \mathbb{Z}[Y_{i,b}(1 + A_{i,bq_i}^{-1})]_{b \in \mathbb{C}^\times}.$$

$\text{Ker } \mathcal{S}_i = \mathbb{Z}[Y_{j,a}^{\pm 1}]_{j \neq i, a \in \mathbb{C}^\times} \otimes \mathbb{Z}[Y_{i,b}(1 + A_{i,bq_i}^{-1})]_{b \in \mathbb{C}^\times}$

(Ex) $\mathfrak{g} = A_2$

$$\begin{aligned} \chi_q(V_{\omega_1,a}) &= Y_{1,a} + Y_{1,aq^2}^{-1} Y_{2,aq} + Y_{2,aq^3}^{-1} \\ &= Y_{1,a}(1 + A_{1,aq}^{-1}) + Y_{2,aq^3}^{-1} = Y_{1,a} + Y_{1,aq^2}^{-1} Y_{2,aq}(1 + A_{2,aq^2}^{-1}) \end{aligned}$$

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- q-character and Weyl group action

The character of Lie group G for $\mathfrak{g}$, χ: Rep $G \to \mathbb{Z}[Y_i^{\pm}]_{i \in S}$
$W(\mathfrak{g}) \curvearrowright \mathbb{Z}[Y_i^{\pm}]_{i \in S}$; $r_i \cdot Y_j = Y_i A_i^{\delta_{i,j}}$; $A_i := \prod_{j \in I} Y_j^{C_{ji}}$.
(Y_i and A_i are multiplicative analogues of ω_i and α_i.)

What is the q-analogue of this Weyl grp action?
Moreover, what is the relation with screening operators?

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What is the q-analogue of this Weyl grp action?
Moreover, what is the relation with screening operators?

q is root of unity: [I 21] (no definition of screening operators)
generic q: [Frenkel-Hernandez 22] (Cf. [I 21])

- Stratification

$\mathbb{C}(Y) := \mathbb{C}(Y_{i,a}; i \in S, a \in \mathbb{C}^\times)$

For $a \in \mathbb{C}^\times$, $\mathbb{C}_a(Y) := \mathbb{C}(Y_{i,aq^{2n+\delta(i)}}; i \in S, n \in D\mathbb{Z})$

$\phi_a : \mathbb{C}_a(Y) \overset{\sim}{\to} \mathbb{C}(y) := \mathbb{C}(y_i(n); i \in S, n \in D\mathbb{Z})$; $\ Y_{i,aq^{2n+\delta(i)}} \mapsto y_i(n)$

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$\rightsquigarrow$ The image of $\mathrm{Im}\chi_q \cap \mathbb{C}_a(Y)$ by ϕ_a:

$$\boldsymbol{\mathcal{Y}}_{\chi_q} := \bigcap_{i \in S} \mathbb{Z}[y_i(n)(1 + X_n^i), y_j(n)^{\pm}; j \neq i, n \in D\mathbb{Z}]$$

$$X_n^i = \phi_a(A^{-1}_{i,aq^{2n+\delta(i)+D_i}}) = \frac{F_i(n)}{y_i(n)y_i(n+D_i)}$$ ($F_i(n)$: a monomial in the $y_j(m)$)

• q-character at root of unity and Weyl group action ($\varepsilon^{2D'm} = 1$)

<u>Fact</u>

(i) [Frenkel-Mukhin 02] χ_ε : Rep $U_\varepsilon^{res}(\hat{\mathfrak{g}}) \to \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in S, a \in \mathbb{C}^\times}$ is the q-character, and (the stratification of) Imχ_ε is

$$\boldsymbol{\mathcal{Y}}_{\chi_\varepsilon} := \bigcap_{i \in S} Z^{(i)}; \; Z^{(i)} := \mathbb{Z}[y_i(n)(1 + X_n^i), y_j(n)^{\pm}; j \neq i, n \in D\mathbb{Z}/D'm\mathbb{Z}]$$

(ii) [I 21] $y_i(n)$ is regarded as variables on $Q'_m(\mathfrak{g})$, and
$X_n^i = \phi_a(A^{-1}_{i, aq^{2n+\delta(i)+D_i}})$ is identified with X-var for $Q'_m(\mathfrak{g})$
(Cf. [Hernandez-Leclerc 16]).

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Def

For $i \in S$, define $r_i \curvearrowright \mathbb{C}(y)_m := \mathbb{C}(y_i(n); v_n^i \in I')$ by .

$$r_i(y_j(n)) = \begin{cases} y_i(n) X^i_{n-D_i} \dfrac{f_X(i, n-2D_i)}{f_X(i, n-D_i)} & j = i, \\ y_j(n) & j \neq i, \end{cases}$$

$I' = \{v_n^i; i \in S, n \in D\mathbb{Z}/D'm\mathbb{Z}\}$: the vertex set of $Q'_m(\mathfrak{g})$

$\mathcal{Y}_{\chi_\varepsilon} := \bigcap_{i \in S} Z^{(i)};\ Z^{(i)} := \mathbb{Z}[y_i(n)(1 + X^i_n), y_j(n)^{\pm}; j \neq i, n \in D\mathbb{Z}/D'm\mathbb{Z}]$

Thm [I 21]

(i) For $i \in S$, the following diagram is commutative. $\left(\beta(X^i_n) = \frac{F_i(n)}{y_i(n) y_i(n+D_i)}\right)$

$$\begin{array}{ccc} \mathbb{C}(\mathbf{X}) & \xrightarrow[\beta]{} & \mathbb{C}(y)_m \\ \Big\downarrow R_i^* & & \Big\downarrow r_i \\ \mathbb{C}(\mathbf{X}) & \xrightarrow[\beta]{} & \mathbb{C}(y)_m \end{array}$$

and r_i ($i \in S$) generate the $W(\mathfrak{g})$-action on $\mathbb{C}(y)_m$.

(ii) It holds that $\mathcal{Y}_{\chi_\varepsilon} \subset \mathbb{C}(y)_m^{W(\mathfrak{g})}$. Especially, $Z^{(i)} \subset \mathbb{C}(y)_m^{r_i}$.

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Thm [I-Yamazaki 22]

Define $\mathbb{C}(Z^{(i)}) := \mathbb{C}(y_i(n)(1+X^i_n), y_j(n)^{\pm}; j\neq i, n\in D\mathbb{Z}/D'm\mathbb{Z})$. For $i\in S$, we have

$$\mathbb{C}(y)_m^{r_i} = \begin{cases} \mathbb{C}(Z^{(i)}) & \frac{D_i}{D}=1 \\ \mathbb{C}(Z^{(i)})(\delta_{i,1}\delta_{i,2}) & \frac{D_i}{D}=2 \\ \mathbb{C}(Z^{(i)})(\delta_{i,1}\delta_{i,2}, \delta_{i,2}\delta_{i,3}) & \frac{D_i}{D}=3. \end{cases}$$

Here, for $1\le\gamma_i\le\frac{D_i}{D}$ we set $\delta_{i,\gamma_i} = \prod_{v^i_n\in P_{i,\gamma_i}} y_i(n) - \prod_{v^i_n\in P_{i,\gamma_i}} \frac{F_i(n)}{y_i(n)}$. $(\delta^2_{i,\gamma_i}\in Z^{(i)})$

• q-character for generic q

Idea

Consider the action $\hat{r}_i$ by replacing $f_X(i,n)$ with its 'infinite' version:

$$\hat{f}_X(i,n) = 1 + \sum_{k=0}^{\infty} X^i_n X^i_{n-D_i} \cdots X^i_{n-kD_i} \in \mathbb{C}[[\mathbf{X}]]$$

What is an appropriate space where the action of $\hat{r}_i$ is well-defined?

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Thm [Frenkel-Hernandez 22]

Π : a certain completion of $\mathbb{Z}[y^{\pm 1}]$ (in a same sense as the 'category O'):
a direct sum of rings of formal power series

$\rightsquigarrow$ 'generalization of the $\hat{r}_i$' generate the Weyl grp action on Π.

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$\mathcal{S}_i; y_j(n) \mapsto \delta_{i,j}\, y_i(n) s_i(n)$: the screening operator

Thm [FH 22]

Set $\hat{f}_X(i,n;h) := h \cdot \hat{f}_X(i,n)$. Then, in the limit $h \to 0$ we have

$$\hat{r}_i = \mathrm{Id} + h \cdot \mathcal{S}_i + O(h^2),$$

where $s_i(n) = -\hat{f}_X(i, n-D_i; h)^{-1}$.

In particular, it holds that $\boldsymbol{\mathcal{Y}}_{\chi_q} = \mathbb{Z}[y^{\pm 1}]^{W(\mathfrak{g})}$.

For q-character, screening operator, Weyl group action, we have

$$\mathrm{Im}\chi_q = \bigcap_{i \in S} \mathrm{Ker}\mathcal{S}_i = \mathbf{Y}^{W(\mathfrak{g})} \quad (\mathbf{Y} = \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in S, a \in \mathbb{C}^\times}).$$

Cluster realization of $W(\mathfrak{g})$

$Q_m(\mathfrak{g})$ $\quad$ $Q_m'(\mathfrak{g})$

- geometric R
  $\mathfrak{g} = A_\ell$ [I-Lam-Pylyavskyy 19]
- q-Painlevé eq.
  $Q_2(A_\ell^{(1)})$
  [Bershtein-Gavrylenko-Marshakov 18]
  [Okubo-Suzuki 22]
  [Masuda-Okubo-Tsuda 21]

- Higher Teichmüller theory
  [Fock-Goncharov 06]
  [Goncharov-Shen 18, 19]
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- Positive rep of $U_q(\mathfrak{g})$
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Thank you!