The BCFW Tiling of the Amplituhedron

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מכון ויצמן למדע
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The Totally Nonnegative Grassmannian

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Lusztig, Postnikov, Rietsch, Fomin Zelevinsky, Marsh, ...

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$$\begin{pmatrix} 1 & 2 & 0 & 0 & 0 & -6 & -5 \\ 0 & 3 & 0 & 4 & 5 & 0 & 0 \end{pmatrix} \in Gr_{\geq}(2,7)$$

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$$P_{24} = 8 \geq 0$$

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1 2 3 4 5 6 7

Totally Nonnegative Grassmannian

Positroids cells: which Plückers are positive?

$S_M = \{ C \in Gr_{\geq}(k,n) : P_I(C) > 0 \text{ iff } I \in M \}$

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$$S_{\{12,13,14\}} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & a & b \end{pmatrix} : a,b \in (0,\infty)^2 \right\}$$

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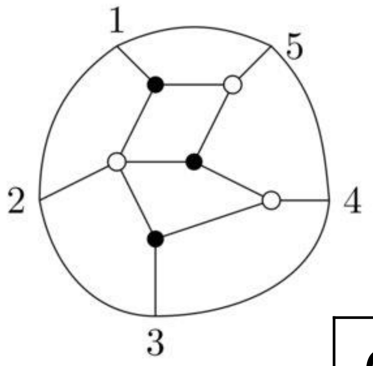
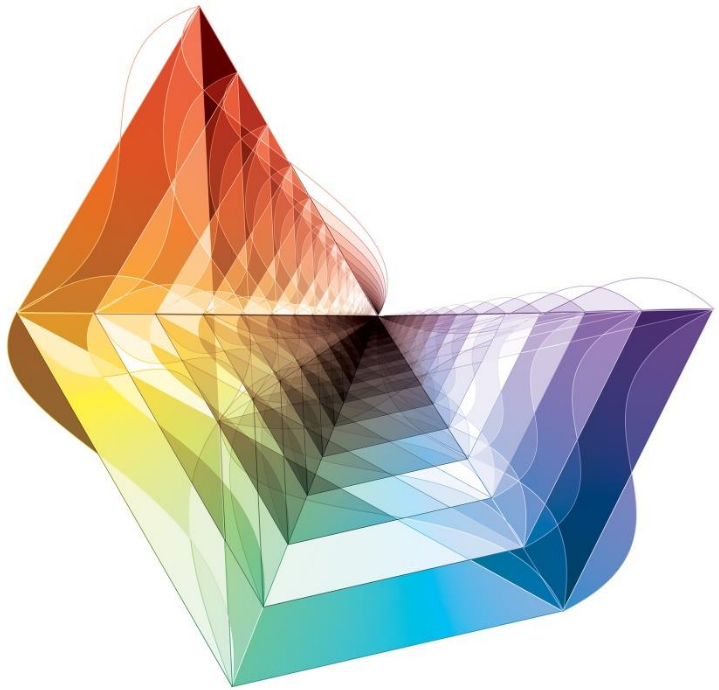
- regular CW complex [Postnikov]
- parametrized by: $(0,\infty)^d$ [$d = \dim S_M$]
- indexed by: matroids / plabic graphs / decorated permutations / (0/1)-tableaux



0	0	0	+
+	0	+	

The Amplituhedron



The Amplituhedron

Arkani-Hamed and Trnka 2013

The Amplituhedron

Arkani-Hamed and Trnka 2013

Fix $Z \in \mathrm{Mat}^{>}_{n \times (k+m)}$

means: all $(k+m)\times(k+m)$ dets > 0

$n \geq k+m$

$$\begin{bmatrix} - & Z_1 & - \\ - & Z_2 & - \\ - & Z_3 & - \\ & \cdots & \\ - & Z_n & - \end{bmatrix}$$

The Amplituhedron

Arkani-Hamed and Trnka 2013

Fix $Z \in Mat^{>}_{n \times (k+m)}$ example: Vandermonde

1	1	1	1	...
1	2	4	8	...
1	3	9	27	...
...				
...				
1	n	n^2	n^3	...

The Amplituhedron

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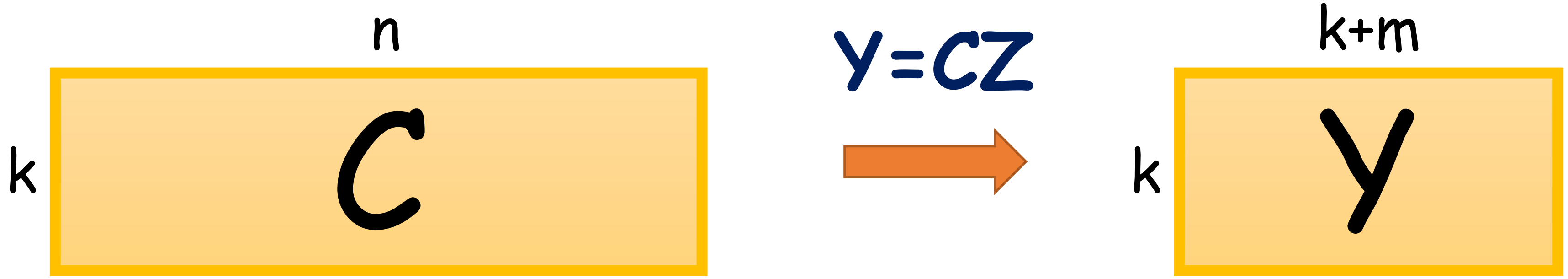
$$Z : Gr_{\geq}(k,n) \rightarrow Gr(k,k+m)$$



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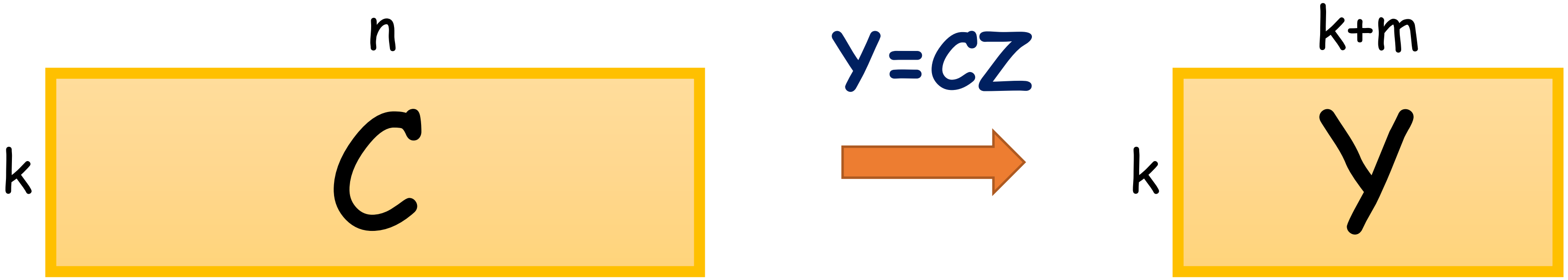

Def: The **Amplituhedron** $A(n,k,m,Z)$ is the image

The Amplituhedron

$A(n,k,m,Z) = \{ CZ : C \in Gr_{\geq}(k,n) \}$ for $Z \in Mat^{>}_{n \times (k+m)}$

- Well-defined in $Gr(k,k+m)$ – no singularities!

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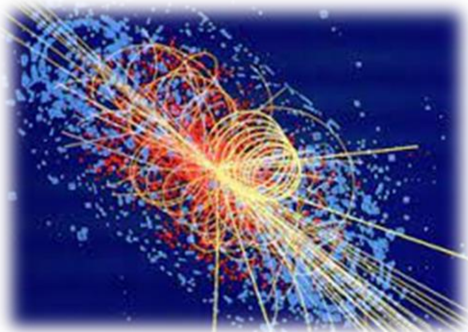

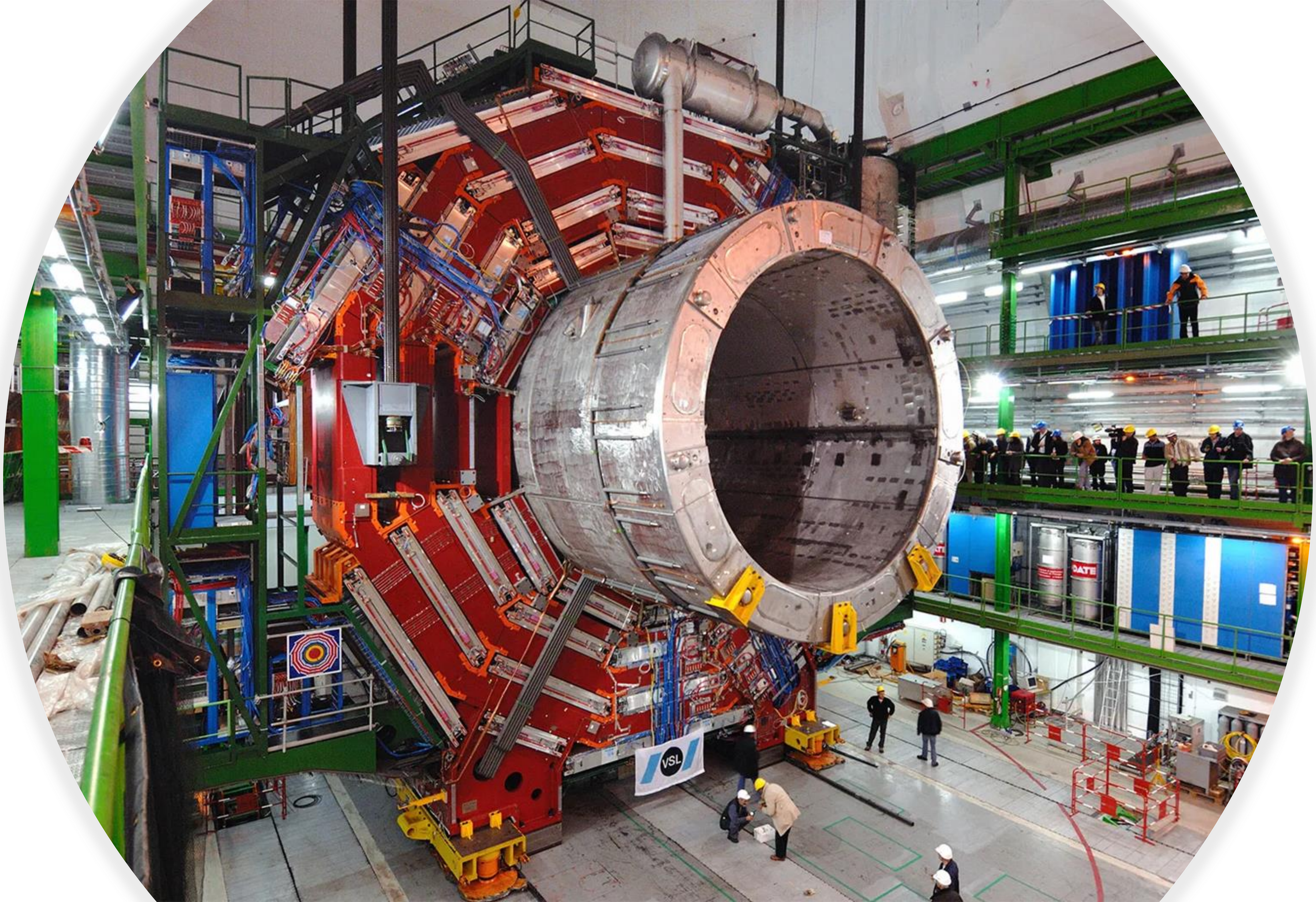
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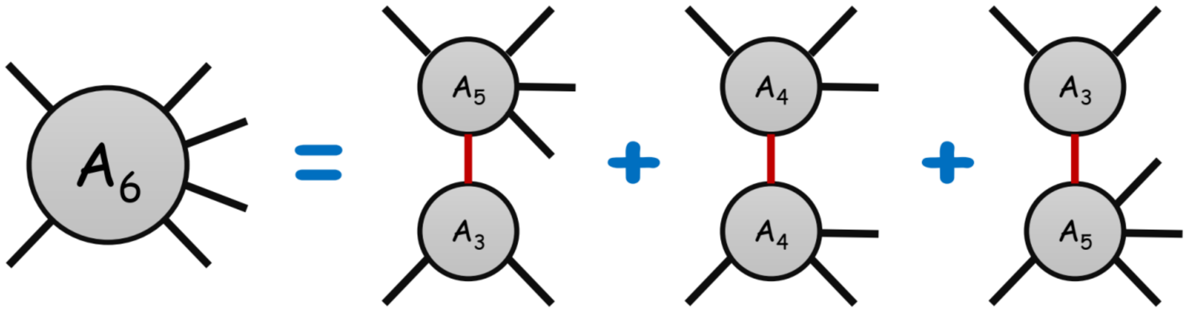
- "Combinatorial structure" should not depend on Z
- "Volume" related to scattering amplitudes (Physics, m=4)
- BCFW recursion (2005)

Britto Cachazo Feng Witten



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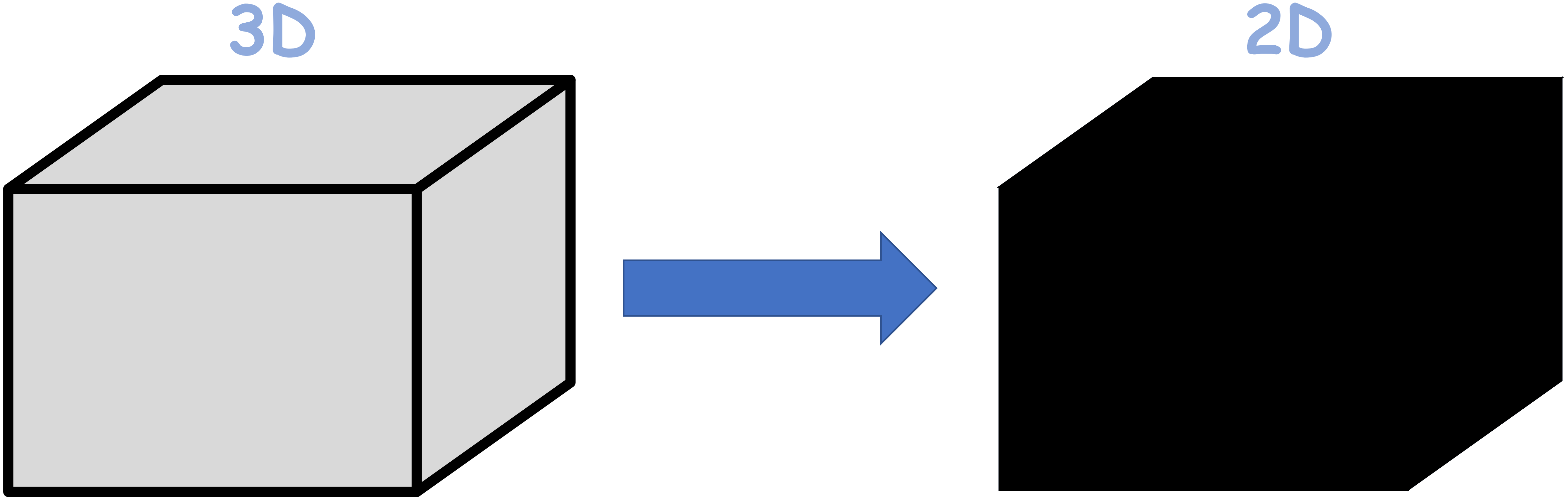
- Compact, connected, full dimensional: km
- "Combinatorial structure" should not depend on Z
- "Volume" related to scattering amplitudes (Physics, m=4)
- BCFW recursion translates to a tiling of $A(n,k,4)$

conjectured by Arkani-Hamed and Trnka 2013

Tiling



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1. $Z : S \to A(n,k,4,Z)$ injective for each $S \in BCFW_{n,k}$

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2. Z-images of $S, S' \in BCFW_{n,k}$ are disjoint for $S \neq S'$
3. Union of Z-images is open dense in $A(n,k,4,Z)$

Known Cases

- $A(n,k=1,m)$ is a cyclic polytope in $\mathbb{RP}^m$
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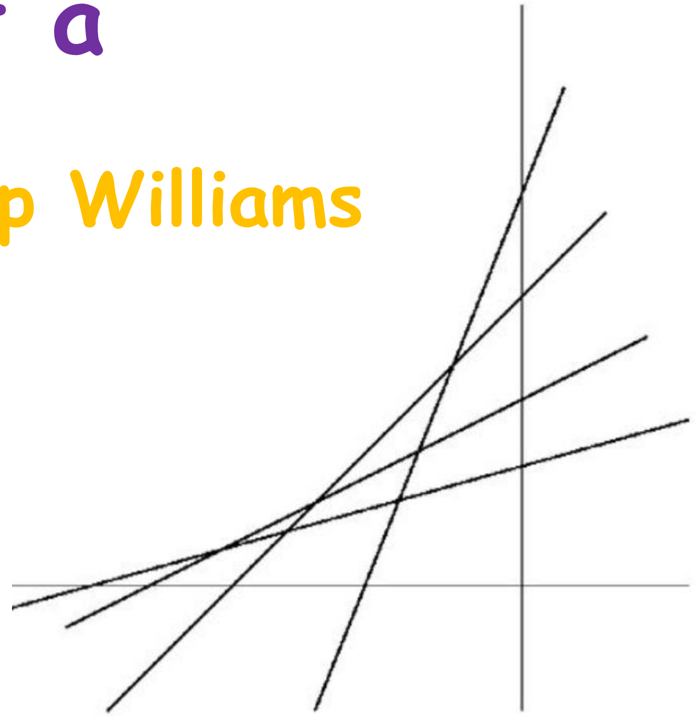
- $A(n,k,m=1)$ is the bounded part of a cyclic hyperplane arrangement Karp Williams



Known Cases

- $A(n,k=1,m)$ is a cyclic polytope in $\mathbb{RP}^m$
- $A(n,k=n-m,m)$ is the totally nonnegative $Gr_{\geq}(k,n)$
- $A(n,k,m=1)$ cyclic hyperplane arrangement KW
- $A(n,k,m)$ and $A(n,n-m-k,m)$ "dual" for even m

Galashin Lam

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- $A(n,k=n-m,m)$ is the totally nonnegative $Gr_{\geq}(k,n)$
- $A(n,k,m=1)$ cyclic hyperplane arrangement KW
- $A(n,k,m=\text{even})$ dual to $A(n,n-m-k,m)$ GL
- $A(n,k,m=2)$ admits "BCFW-type" tiling Bao He

more tilings, map to hypersimplex $\Delta_{k+1,n}$, …

Lukowski Parisi Spradlin Volovich, Parisi Sherman-Bennett Williams

Known Cases

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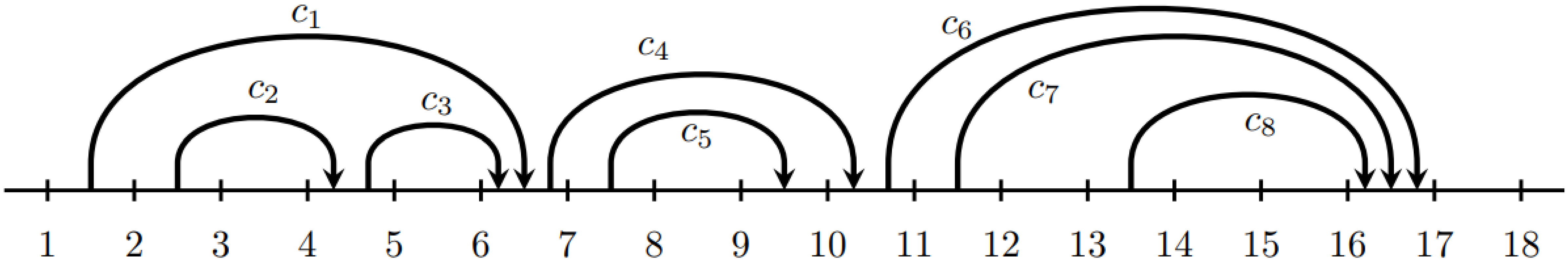
- $A(n,k,m=1)$ cyclic hyperplane arrangement KW
- $A(n,k,m=\text{even})$ dual to $A(n,n-m-k,m)$ GL
- $A(n,k,m=2)$ tilings, hypersimplex BH, LPSV, PSW
- $A(n,k,m=4)$ conjectured $BCFW_{n,k}$ "domino form"

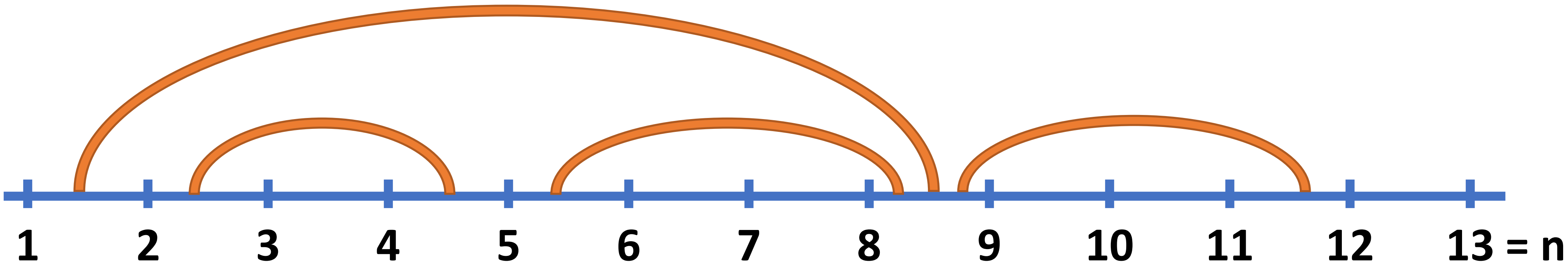
Karp Williams Zhang + Thomas

The BCFW Tiles



Chord Diagrams

Let $CD_{n,k}$ be the set of all chord diagrams,



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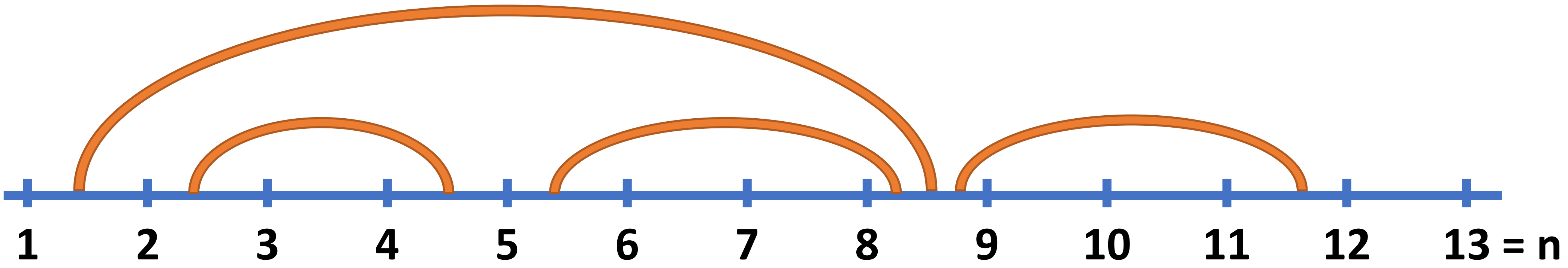
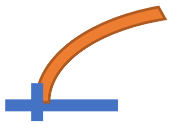
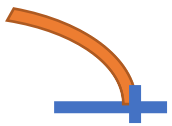


Rules: No  No 

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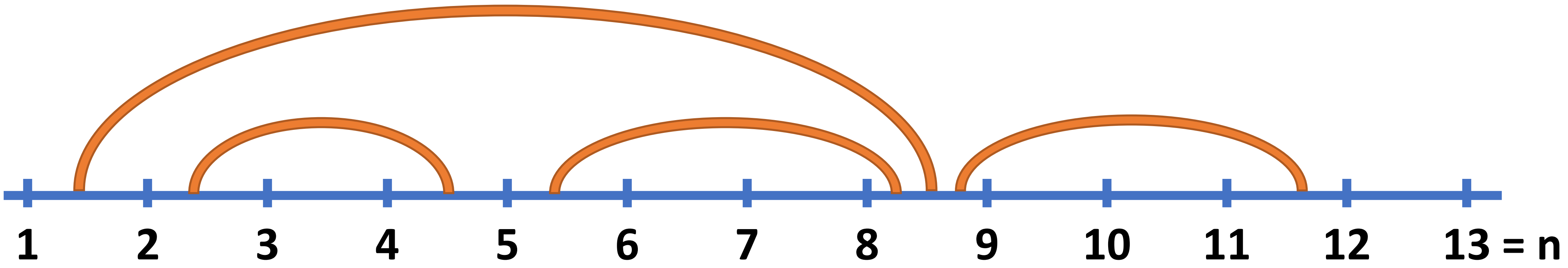
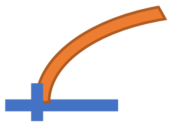
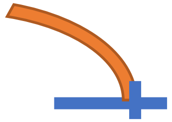
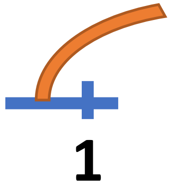
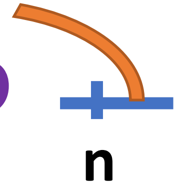
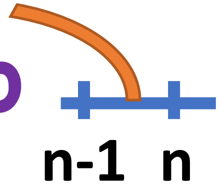


Rules: No  No  No 

No 

No 

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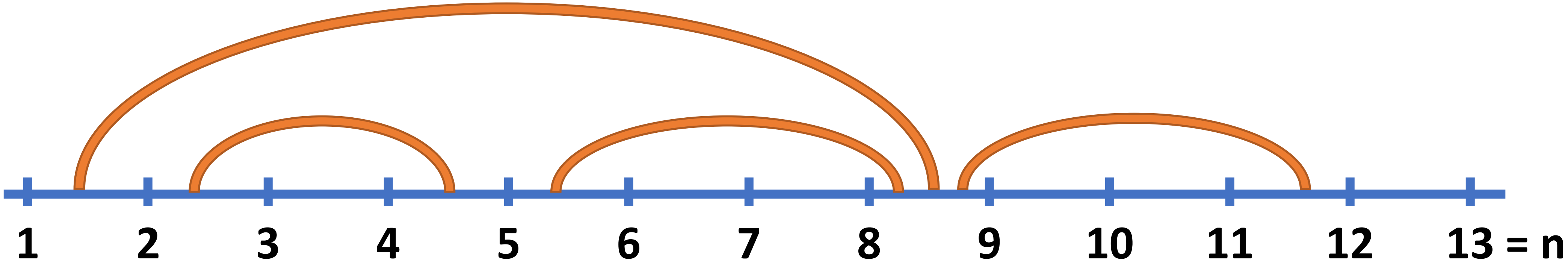
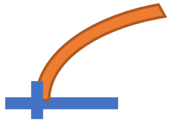
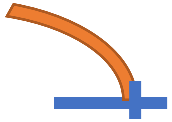
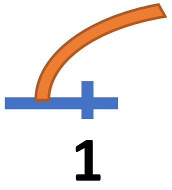
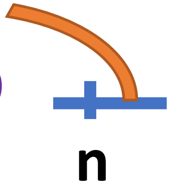
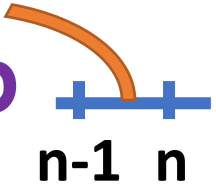
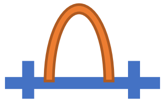
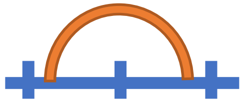


Rules: No  No  No 

No 

No 

No  No 

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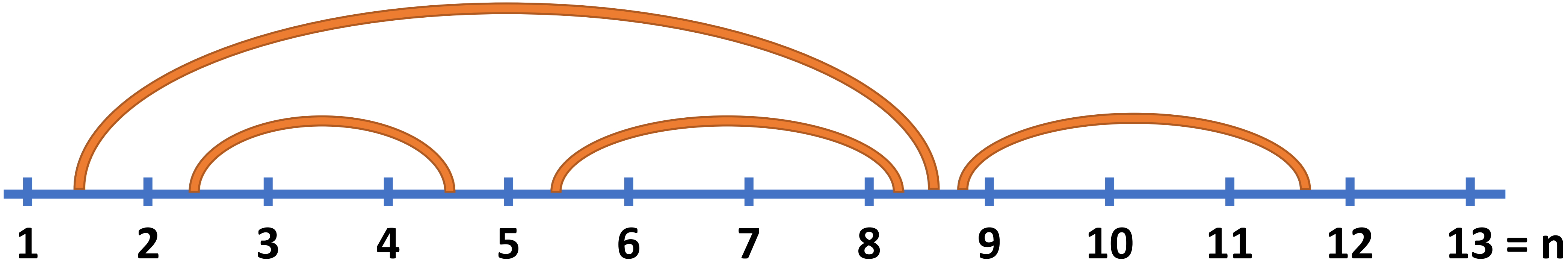
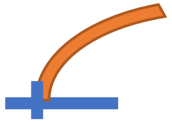
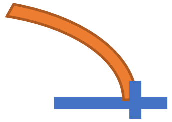
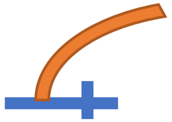
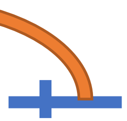
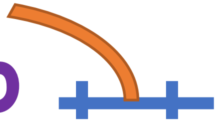
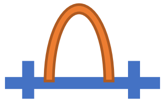
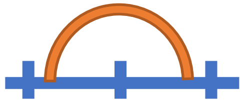
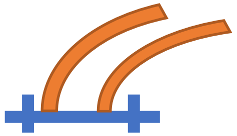


Rules: No  No  No  1 No  n No  n-1 n

No  No  No 

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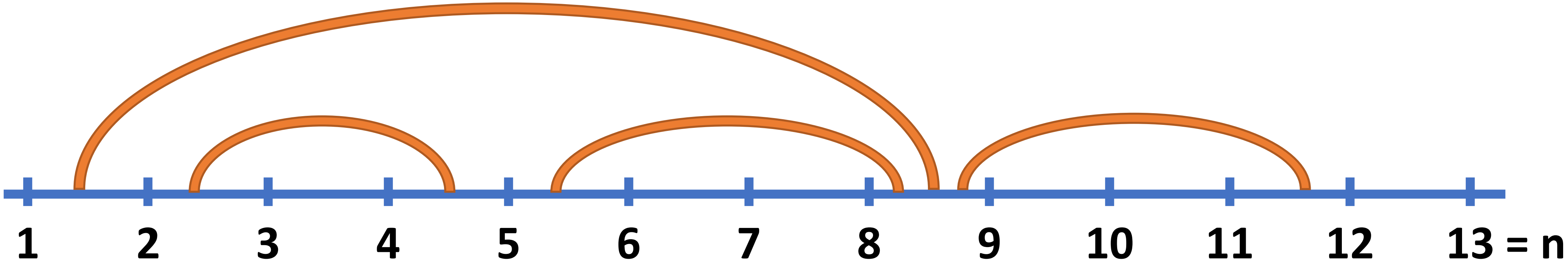
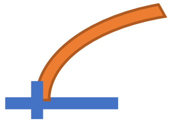
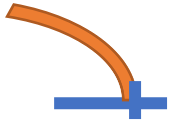
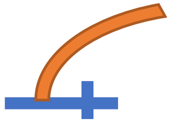
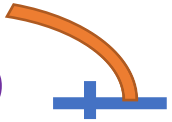
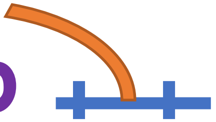
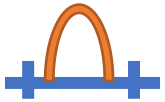
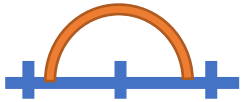
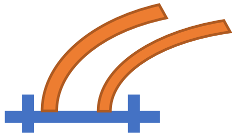
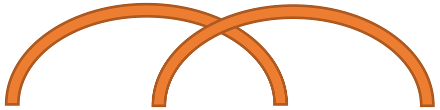


Rules: No  No  No 

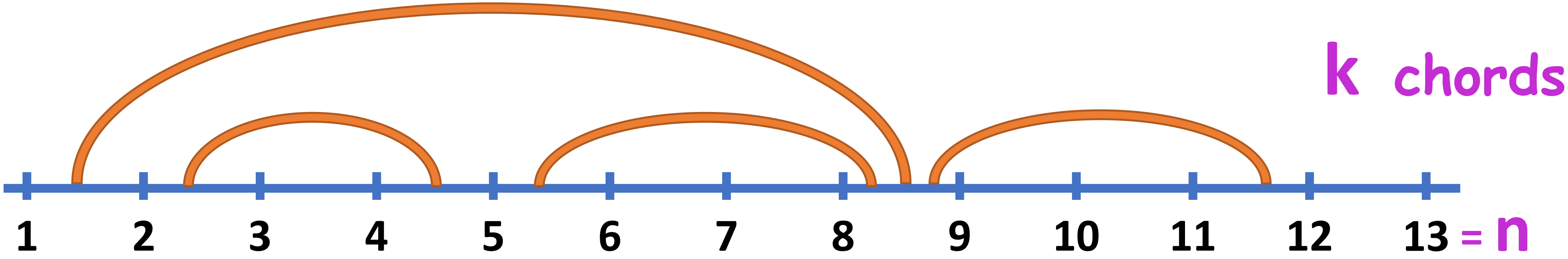
No 

No 

No  No  No  No 

Chord Diagrams

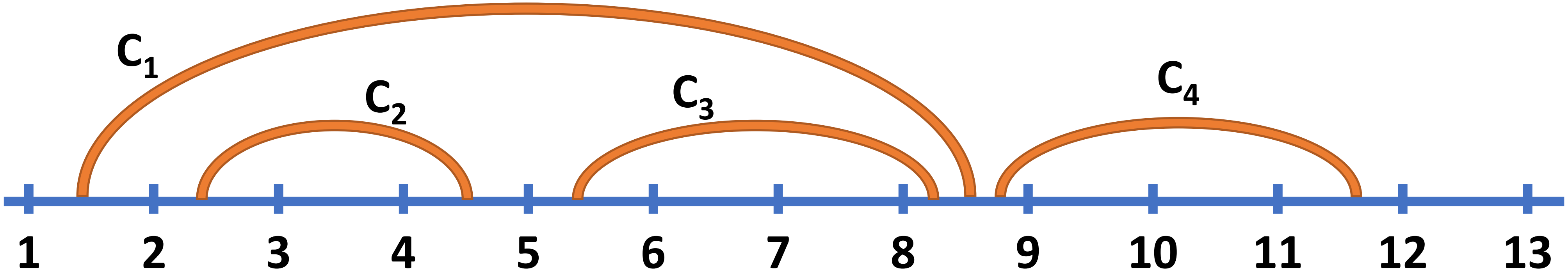
Let $CD_{n,k}$ be the set of all chord diagrams,



$$|CD_{n,k}| = \frac{1}{k+1}\binom{n-3}{k}\binom{n-4}{k} = |BCFW_{n,k}|$$

Defining $BCFW_{n,k}$ cells

Domino matrices of $CD_{n,k}$ give $BCFW_{n,k}$ [ELT]



α_1	β_1						γ_1	δ_1				
	α_2	β_2	γ_2	δ_2								
			α_3	β_3			γ_3	δ_3				
							α_4	β_4		γ_4	δ_4	

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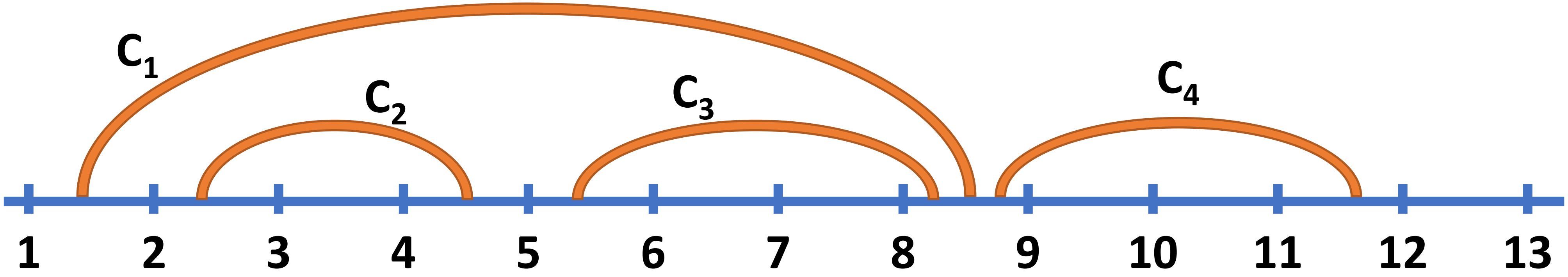

α_1	β_1						γ_1	δ_1				
	α_2	β_2	γ_2	δ_2								
				α_3	β_3		γ_3	δ_3				
							α_4	β_4		γ_4	δ_4	

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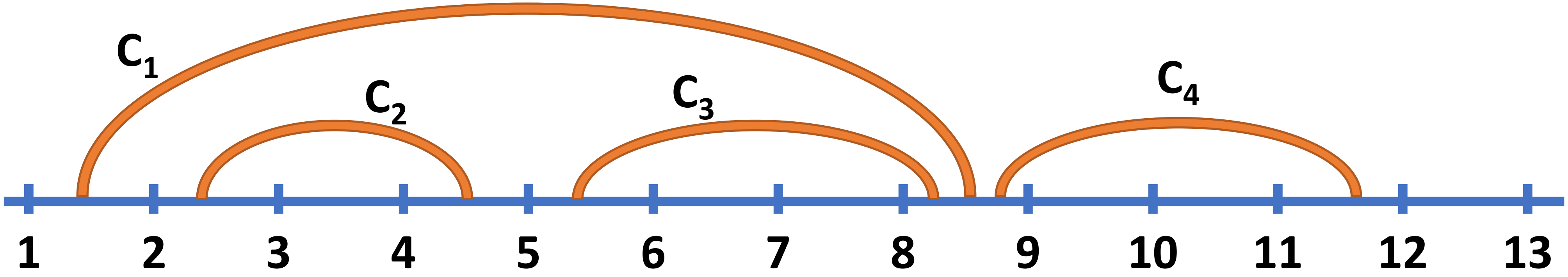
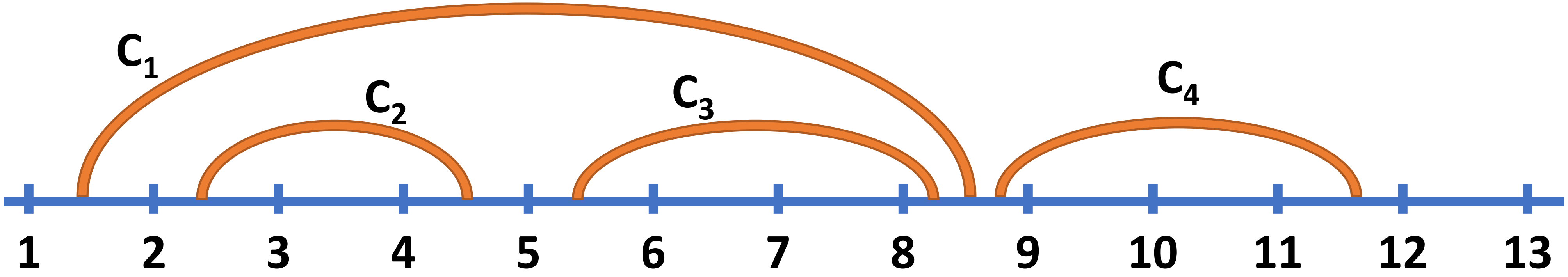


α_1	β_1						γ_1	δ_1				ϵ_1
	α_2	β_2	γ_2	δ_2								
				α_3	β_3		γ_3	δ_3				
							α_4	β_4		γ_4	δ_4	ϵ_4

Defining $BCFW_{n,k}$ cells

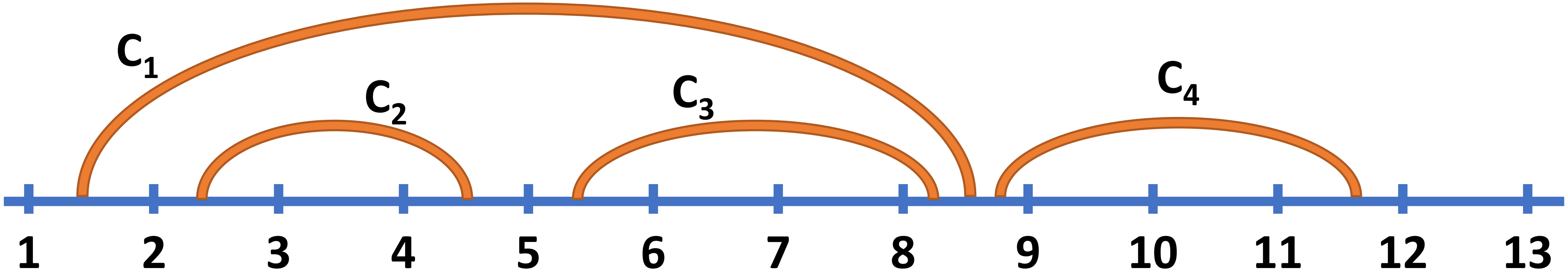
Domino matrices of $CD_{n,k}$ give $BCFW_{n,k}$ [ELT]



α_1	β_1						γ_1	δ_1				ϵ_1
	α_2	β_2	γ_2	δ_2								
$\epsilon_3\alpha_1$	$\epsilon_3\beta_1$			α_3	β_3		γ_3	δ_3				
							α_4	β_4		γ_4	δ_4	ϵ_4

Defining $BCFW_{n,k}$ cells

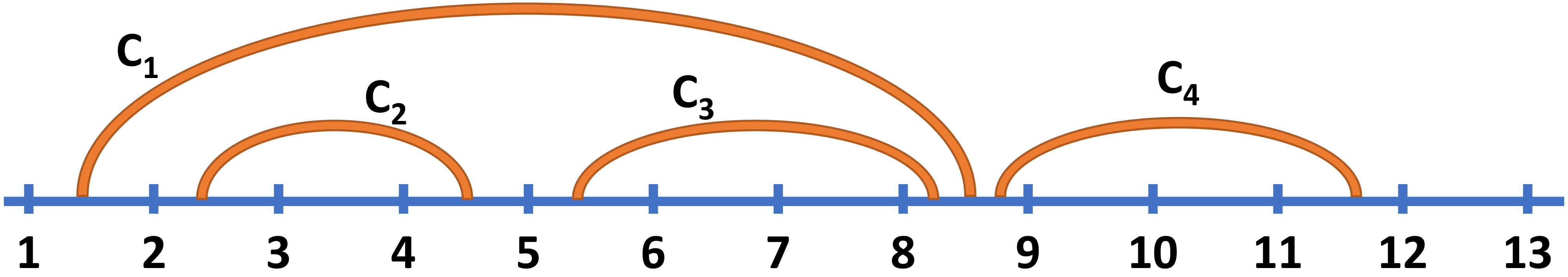
Domino matrices of $CD_{n,k}$ give $BCFW_{n,k}$ [ELT]



α_1	β_1						γ_1	δ_1				ε_1
$\varepsilon_2\alpha_1$	$\varepsilon_2\beta_1+\alpha_2$	β_2	γ_2	δ_2								
$\varepsilon_3\alpha_1$	$\varepsilon_3\beta_1$			α_3	β_3		γ_3	δ_3				
							α_4	β_4		γ_4	δ_4	ε_4

Defining $BCFW_{n,k}$ cells

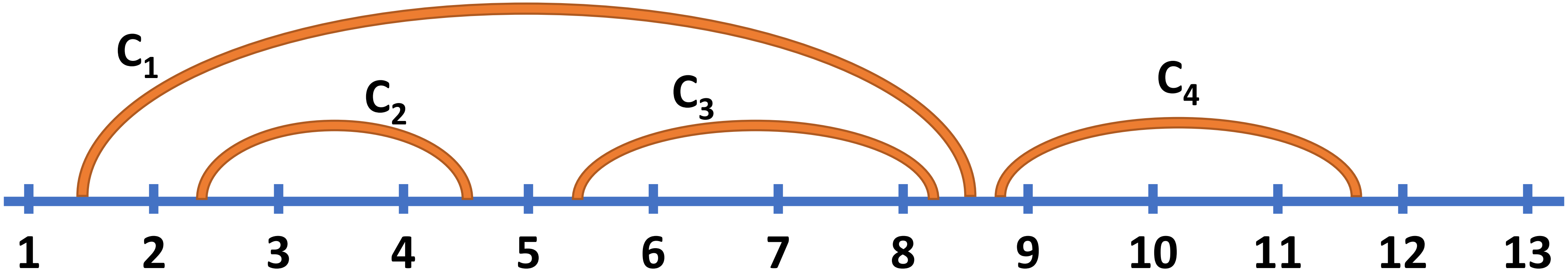
Domino matrices of $CD_{n,k}$ give $BCFW_{n,k}$ [ELT]



α_1	β_1	0	0	0	0	0	γ_1	δ_1	0	0	0	ε_1
$\varepsilon_2\alpha_1$	$\varepsilon_2\beta_1+\alpha_2$	β_2	γ_2	δ_2	0	0	0	0	0	0	0	0
$\varepsilon_3\alpha_1$	$\varepsilon_3\beta_1$	0	0	α_3	β_3	0	γ_3	δ_3	0	0	0	0
0	0	0	0	0	0	0	α_4	β_4	0	γ_4	δ_4	ε_4

Defining $BCFW_{n,k}$ cells

Domino matrices of $CD_{n,k}$ give $BCFW_{n,k}$ [ELT]

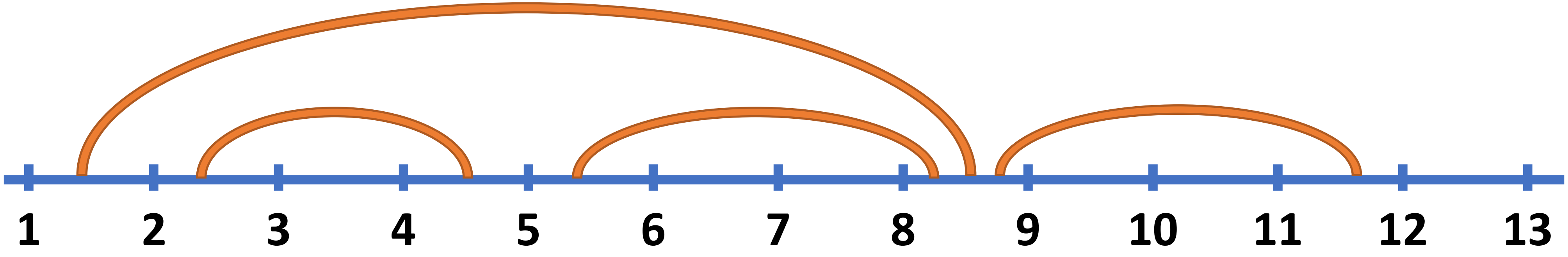


α_1	β_1	0	0	0	0	0	γ_1	δ_1	0	0	0	ε_1
$\varepsilon_2\alpha_1$	$\varepsilon_2\beta_1+\alpha_2$	β_2	γ_2	δ_2	0	0	0	0	0	0	0	0
$\varepsilon_3\alpha_1$	$\varepsilon_3\beta_1$	0	0	α_3	β_3	0	γ_3	δ_3	0	0	0	0
0	0	0	0	0	0	0	α_4	β_4	0	γ_4	δ_4	ε_4

Sign rules: α_i β_i are +. γ_i δ_i ε_i are ± depending on chord count. $\delta_3/\gamma_3 < \delta_1/\gamma_1 < \beta_4/\alpha_4$

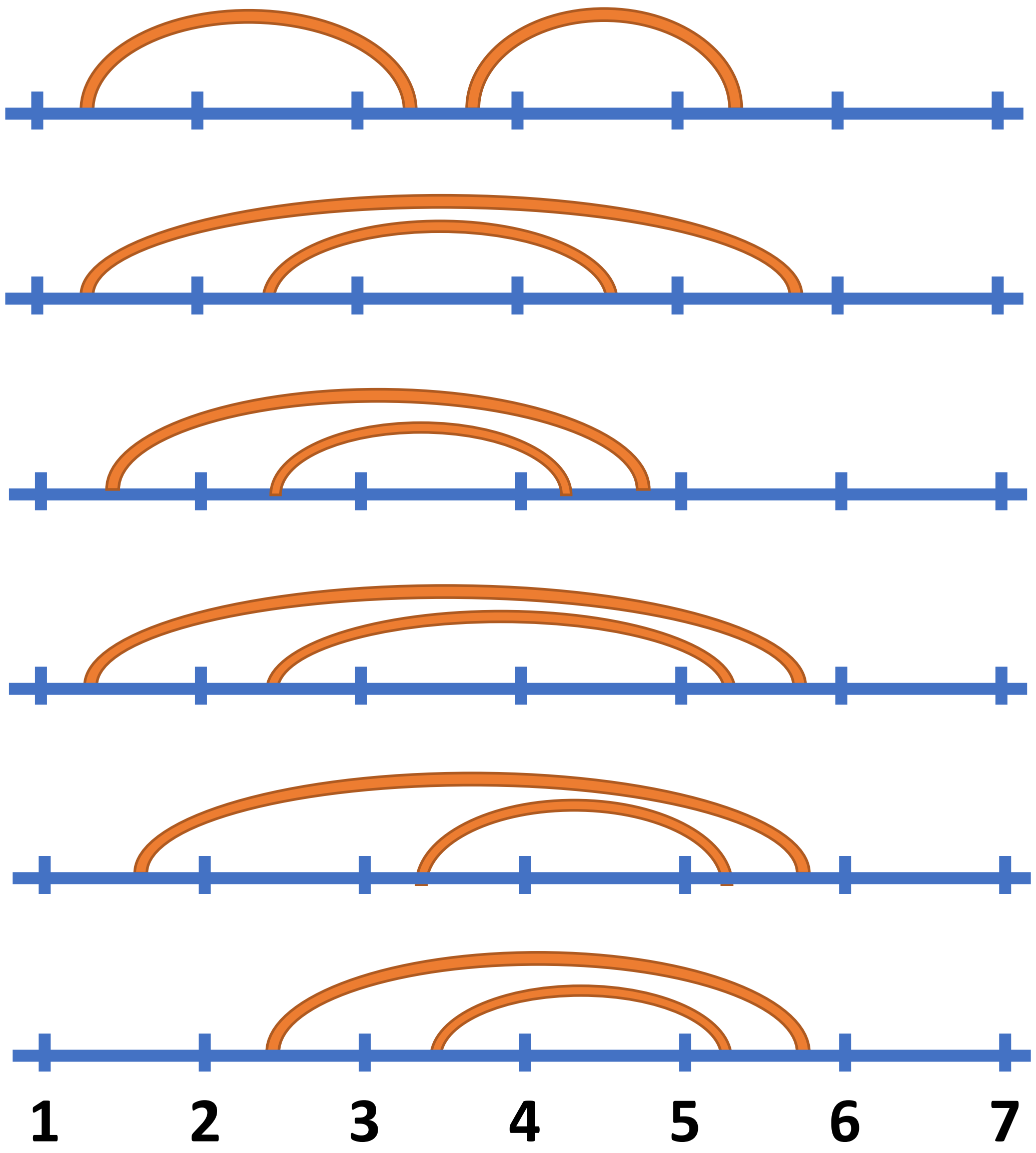
The Domino Theorem [ELT]

The domino matrix of a chord diagram restricted by the sign rules uniquely parametrizes a 4k-dim positroid cell, up to rescaling rows. These are exactly the BCFW cells as previously defined.



α_1	β_1	0	0	0	0	0	γ_1	δ_1	0	0	0	ε_1
$\varepsilon_2\alpha_1$	$\varepsilon_2\beta_1+\alpha_2$	β_2	γ_2	δ_2	0	0	0	0	0	0	0	0
$\varepsilon_3\alpha_1$	$\varepsilon_3\beta_1$	0	0	α_3	β_3	0	γ_3	δ_3	0	0	0	0
0	0	0	0	0	0	0	α_4	β_4	0	γ_4	δ_4	ε_4

Tiling of A(n=7,k=2,m=4)



α_1	β_1	γ_1	δ_1	0	0	ε_1
0	0	α_2	β_2	γ_2	δ_2	ε_2

α_1	β_1	0	0	γ_1	δ_1	ε_1
$\varepsilon_2\alpha_1$	$\varepsilon_2\beta_1+\alpha_2$	β_2	γ_2	δ_2	0	0

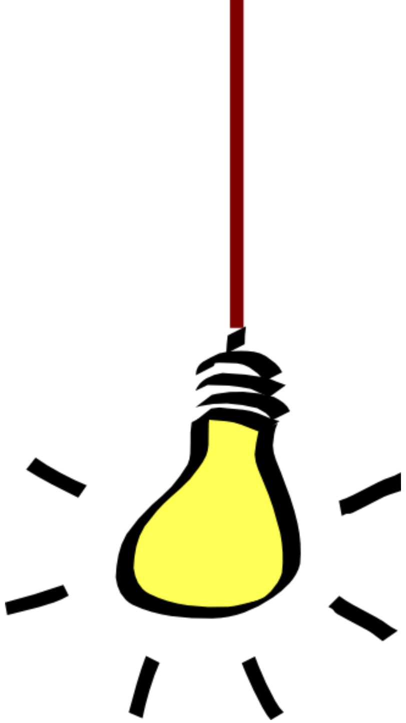
α_1	β_1	0	γ_1	δ_1	0	ε_1
$\varepsilon_2\alpha_1$	$\varepsilon_2\beta_1+\alpha_2$	β_2	γ_2	δ_2	0	0

α_1	β_1	0	0	γ_1	δ_1	ε_1
$\varepsilon_2\alpha_1$	$\varepsilon_2\beta_1+\alpha_2$	β_2	0	γ_2	δ_2	0

α_1	β_1	0	0	γ_1	δ_1	ε_1
$\varepsilon_2\alpha_1$	$\varepsilon_2\beta_1$	α_2	β_2	γ_2	δ_2	0

0	α_1	β_1	0	γ_1	δ_1	ε_1
0	$\varepsilon_2\alpha_1$	$\varepsilon_2\beta_1+\alpha_2$	β_2	γ_2	δ_2	0

BCFW Tiling Proof Ideas



Twistor Coordinates

$C \in Gr_{\geq}(k,n)$

$Z \in Mat^{>}_{n \times (k+4)}$

$Y = CZ \in \mathcal{A}(n,k,4,Z) \subset Gr(k,k+4)$

Twistor:

$$\langle a\ b\ c\ d \rangle = \begin{vmatrix} Y \\ Z_a \\ Z_b \\ Z_c \\ Z_d \end{vmatrix}$$

Arkani-Hamed Thomas Trnka

Injectivity

1. $Z : S \to A(n,k,4,Z)$ injective for each $S \in BCFW_{n,k}$

Proof Idea: construct a preimage

Injectivity

1. $Z : S \rightarrow A(n,k,4,Z)$ injective for each $S \in BCFW_{n,k}$

Proof Idea: construct a preimage

Solve each **row of 5** by twistors:

α_1	β_1	γ_1	δ_1	0	0	ε_1
0	0	α_2	β_2	γ_2	δ_2	ε_2
1	2	3	4	5	6	7

Injectivity

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Solve each row of 5 by twistors:

α_1	β_1	γ_1	δ_1	0	0	ϵ_1
0	0	α_2	β_2	γ_2	δ_2	ϵ_2

$\langle 2347\rangle$	$-\langle 1347\rangle$	$\langle 1247\rangle$	$-\langle 1237\rangle$	0	0	$-\langle 1234\rangle$
0	0	$\langle 4567\rangle$	$-\langle 3567\rangle$	$\langle 3467\rangle$	$-\langle 3457\rangle$	$\langle 3456\rangle$
1	2	3	4	5	6	7

Injectivity

Solve each row of 6 after its parent:

α_1	β_1	0	0	γ_1	δ_1	ε_1
$\varepsilon_2\alpha_1$	$\varepsilon_2\beta_1$	α_2	β_2	γ_2	δ_2	0

Injectivity

Solve each **row of 6** after its parent:

Solve row 1

α_1	β_1	0	0	γ_1	δ_1	ε_1
$\varepsilon_2\alpha_1$	$\varepsilon_2\beta_1$	α_2	β_2	γ_2	δ_2	0

$-\langle 2567\rangle$	$\langle 1567\rangle$	0	0	$-\langle 1267\rangle$	$\langle 1257\rangle$	$-\langle 1256\rangle$
$\varepsilon_2\alpha_1$	$\varepsilon_2\beta_1$	α_2	β_2	γ_2	δ_2	0

Injectivity

Solve each row of 6 after its parent:

Solve row 1

$X = \alpha_1 1 + \beta_1 2$

α_1	β_1	0	0	γ_1	δ_1	ε_1
$\varepsilon_2\alpha_1$	$\varepsilon_2\beta_1$	α_2	β_2	γ_2	δ_2	0

$-\langle 2567 \rangle$	$\langle 1567 \rangle$	0	0	$-\langle 1267 \rangle$	$\langle 1257 \rangle$	$-\langle 1256 \rangle$
$\varepsilon_2\alpha_1$	$\varepsilon_2\beta_1$	α_2	β_2	γ_2	δ_2	0

$-\langle 2567 \rangle$	$\langle 1567 \rangle$	0	0	$-\langle 1267 \rangle$	$\langle 1257 \rangle$	$-\langle 1256 \rangle$
$\alpha_1\langle 3456 \rangle$	$\beta_1\langle 3456 \rangle$	$-\langle X456 \rangle$	$\langle X356 \rangle$	$-\langle X346 \rangle$	$\langle X345 \rangle$	0

Injectivity

Solve each row of 6 after its parent:

Solve row 1

$X = \alpha_1 1 + \beta_1 2$

α_1	β_1	0	0	γ_1	δ_1	ε_1
$\varepsilon_2\alpha_1$	$\varepsilon_2\beta_1$	α_2	β_2	γ_2	δ_2	0

-<2567>	<1567>	0	0	-<1267>	<1257>	-<1256>
$\varepsilon_2\alpha_1$	$\varepsilon_2\beta_1$	α_2	β_2	γ_2	δ_2	0

-<2567>	<1567>	0	0	-<1267>	<1257>	-<1256>
α_1<3456>	β_1<3456>	-<X456>	<X356>	-<X346>	<X345>	0

||

<2567> <1346> - <1567> <2346>

Injectivity

Solve each row of 6 after its parent:

α_1	β_1	0	0	γ_1	δ_1	ε_1
$\varepsilon_2\alpha_1$	$\varepsilon_2\beta_1$	α_2	β_2	γ_2	δ_2	0

Solve row 1

$-\langle 2567\rangle$	$\langle 1567\rangle$	0	0	$-\langle 1267\rangle$	$\langle 1257\rangle$	$-\langle 1256\rangle$
$\varepsilon_2\alpha_1$	$\varepsilon_2\beta_1$	α_2	β_2	γ_2	δ_2	0

$X = \alpha_1 1 + \beta_1 2$

$-\langle 2567\rangle$	$\langle 1567\rangle$	0	0	$-\langle 1267\rangle$	$\langle 1257\rangle$	$-\langle 1256\rangle$
$\alpha_1\langle 3456\rangle$	$\beta_1\langle 3456\rangle$	$-\langle X456\rangle$	$\langle X356\rangle$	**$-\langle X346\rangle$**	$\langle X345\rangle$	0

$||$

functionary → $\langle 2567\rangle\langle 1346\rangle - \langle 1567\rangle\langle 2346\rangle$

Injectivity

Solve each row of 6 after its parent:

Solve row 1

α_1	β_1	0	0	γ_1	δ_1	ε_1
$\varepsilon_2\alpha_1$	$\varepsilon_2\beta_1$	α_2	β_2	γ_2	δ_2	0

$X = \alpha_1 1 + \beta_1 2$

$-\langle 2567\rangle$	$\langle 1567\rangle$	0	0	$-\langle 1267\rangle$	$\langle 1257\rangle$	$-\langle 1256\rangle$
$\varepsilon_2\alpha_1$	$\varepsilon_2\beta_1$	α_2	β_2	γ_2	δ_2	0

$-\langle 2567\rangle$	$\langle 1567\rangle$	0	0	$-\langle 1267\rangle$	$\langle 1257\rangle$	$-\langle 1256\rangle$
$\alpha_1\langle 3456\rangle$	$\beta_1\langle 3456\rangle$	$-\langle X456\rangle$	$\langle X356\rangle$	$-\langle X346\rangle$	$\langle X345\rangle$	0

$$||$$

$$\langle 567\text{-}12\text{-}346\rangle = \langle 2567\rangle\langle 1346\rangle - \langle 1567\rangle\langle 2346\rangle$$

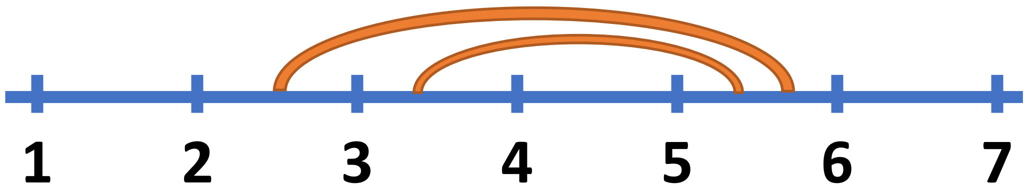
Separation

2. Z-images of $S, S' \in BCFW_{n,k}$ are disjoint for $S \neq S'$

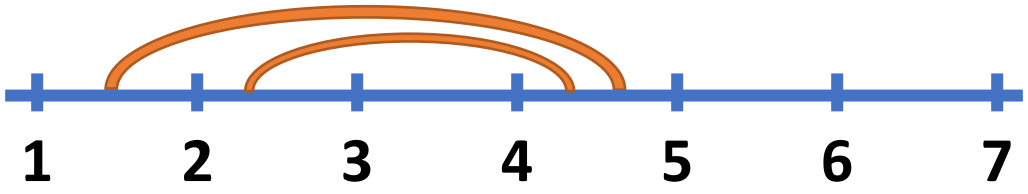
Proof Idea: construct twistor or functionary positive on S and negative on S' by induction on chord diagram structure

Separation

Case I: Last chords with different ends



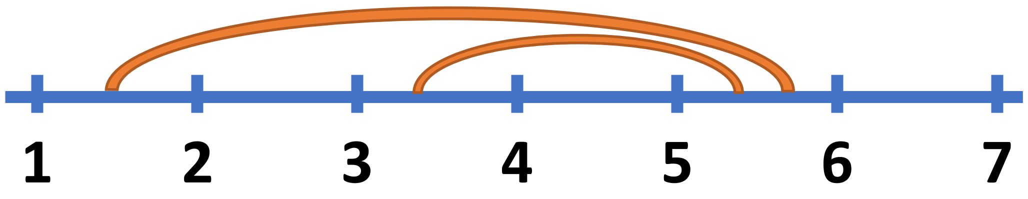
$\langle 2357 \rangle < 0$



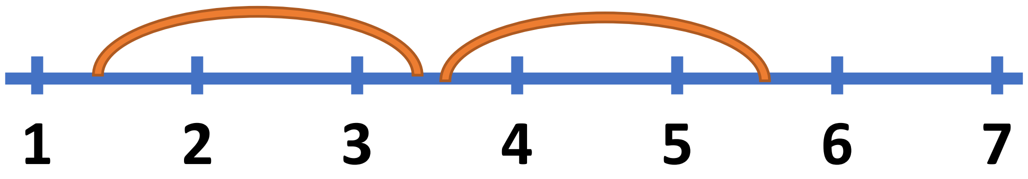
$\langle 2357 \rangle > 0$

Separation

Case II: Same last ends, different starts



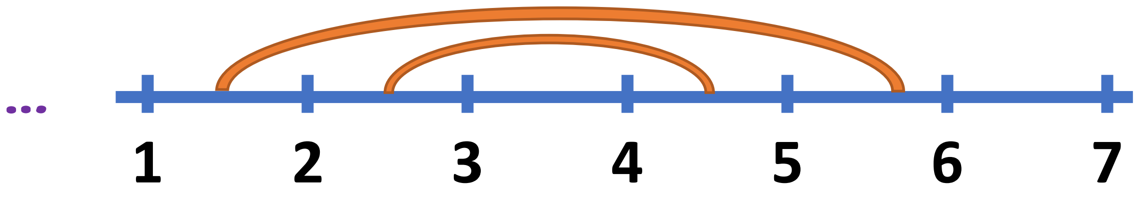
$\langle 712\text{-}34\text{-}567 \rangle < 0$



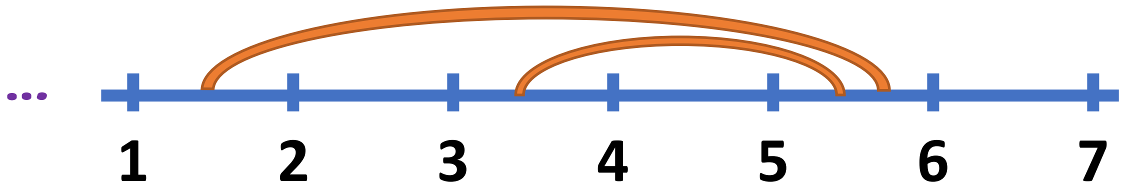
$\langle 712\text{-}34\text{-}567 \rangle > 0$

Separation

Case III: Same last chord, diff subdiagram



<345-12-567> > 0



<345-12-567> < 0

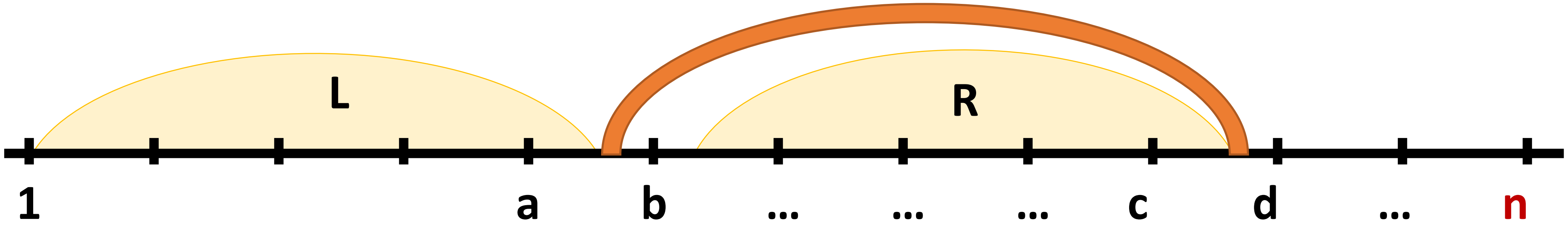
Separation

I cd>c'd': Take <abcn>

II cd=c'd', ab>a'b': Take <na'b'-ab-cdn>

III abcd=a'b'c'd': Promote subdiagram's separators

- R≠R': d to <abcn>d-<abdn>c, n to <abcd>n-<abcn>d+<abdn>c
- L≠L': b to <acdn>b-<bcdn>a



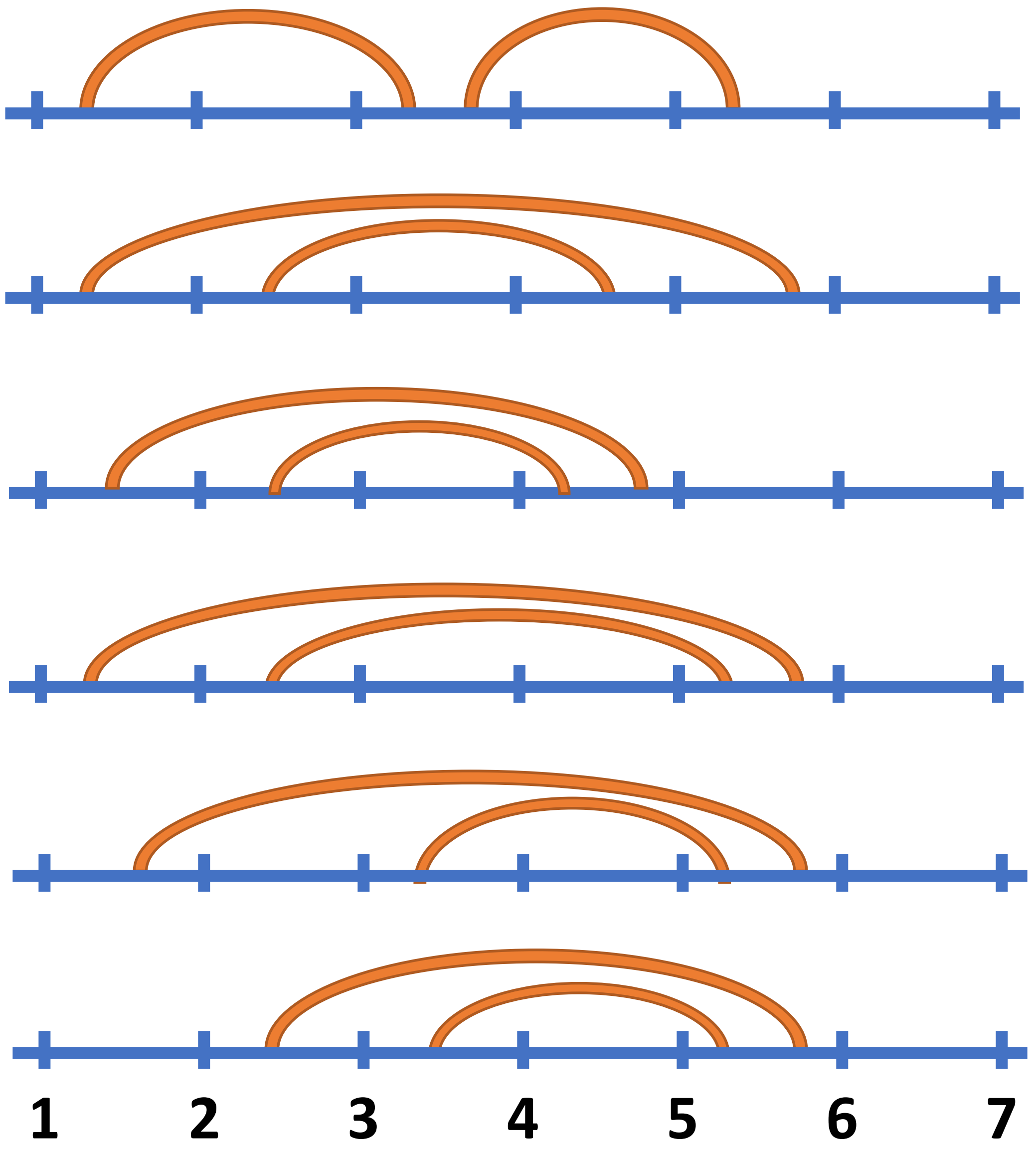
Surjectivity

3. Union of Z-images is open dense in $A(n,k,4,Z)$

<u>Proof Idea:</u> You cannot escape $BCFW_{n,k}$

- Identify codim-1 boundaries of cells
- Each one belongs to 2 cells or $\partial A(n,k,4)$
- Connectivity via transversal path in interior

Tiling of $A(7,2,4)$



α_1	β_1	γ_1	δ_1	0	0	ε_1
0	0	α_2	β_2	γ_2	δ_2	ε_2

α_1	β_1	0	0	γ_1	δ_1	ε_1
$\varepsilon_2\alpha_1$	$\varepsilon_2\beta_1+\alpha_2$	β_2	γ_2	δ_2	0	0

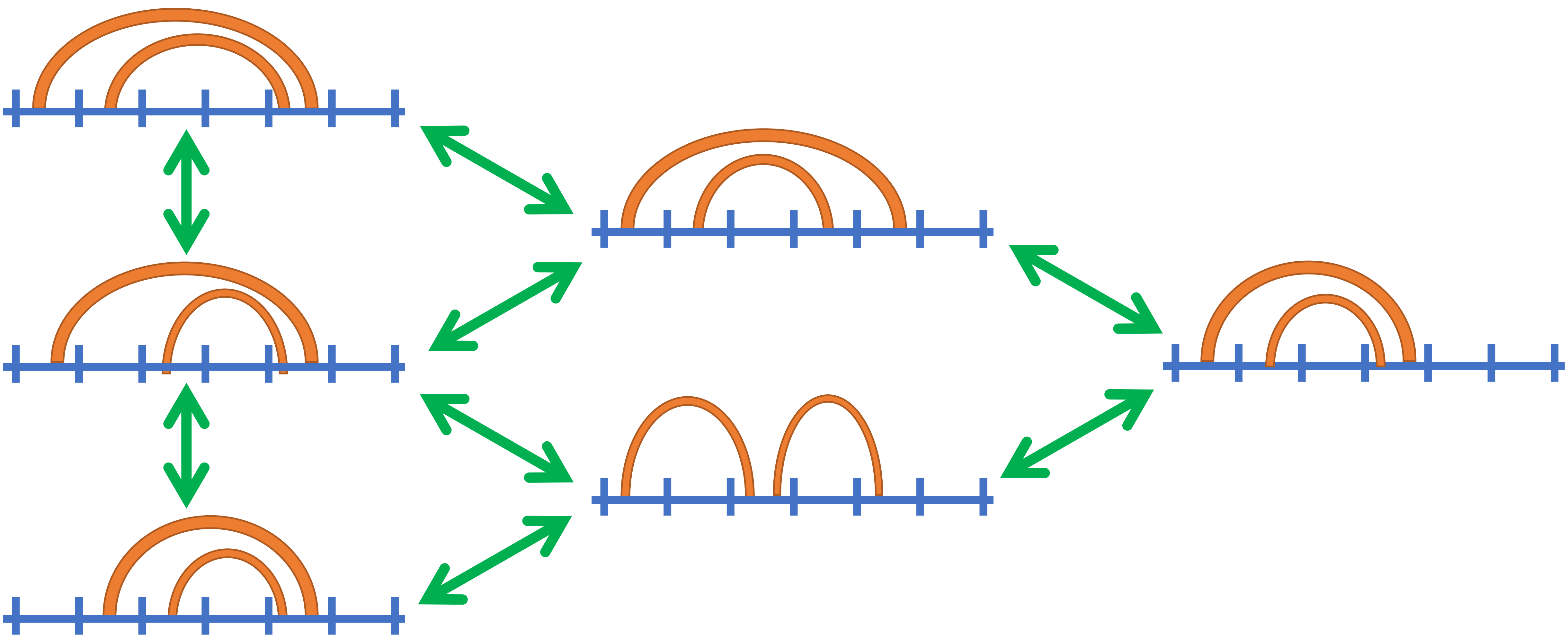
α_1	β_1	0	γ_1	δ_1	0	ε_1
$\varepsilon_2\alpha_1$	$\varepsilon_2\beta_1+\alpha_2$	β_2	γ_2	δ_2	0	0

α_1	β_1	0	0	γ_1	δ_1	ε_1
$\varepsilon_2\alpha_1$	$\varepsilon_2\beta_1+\alpha_2$	β_2	0	γ_2	δ_2	0

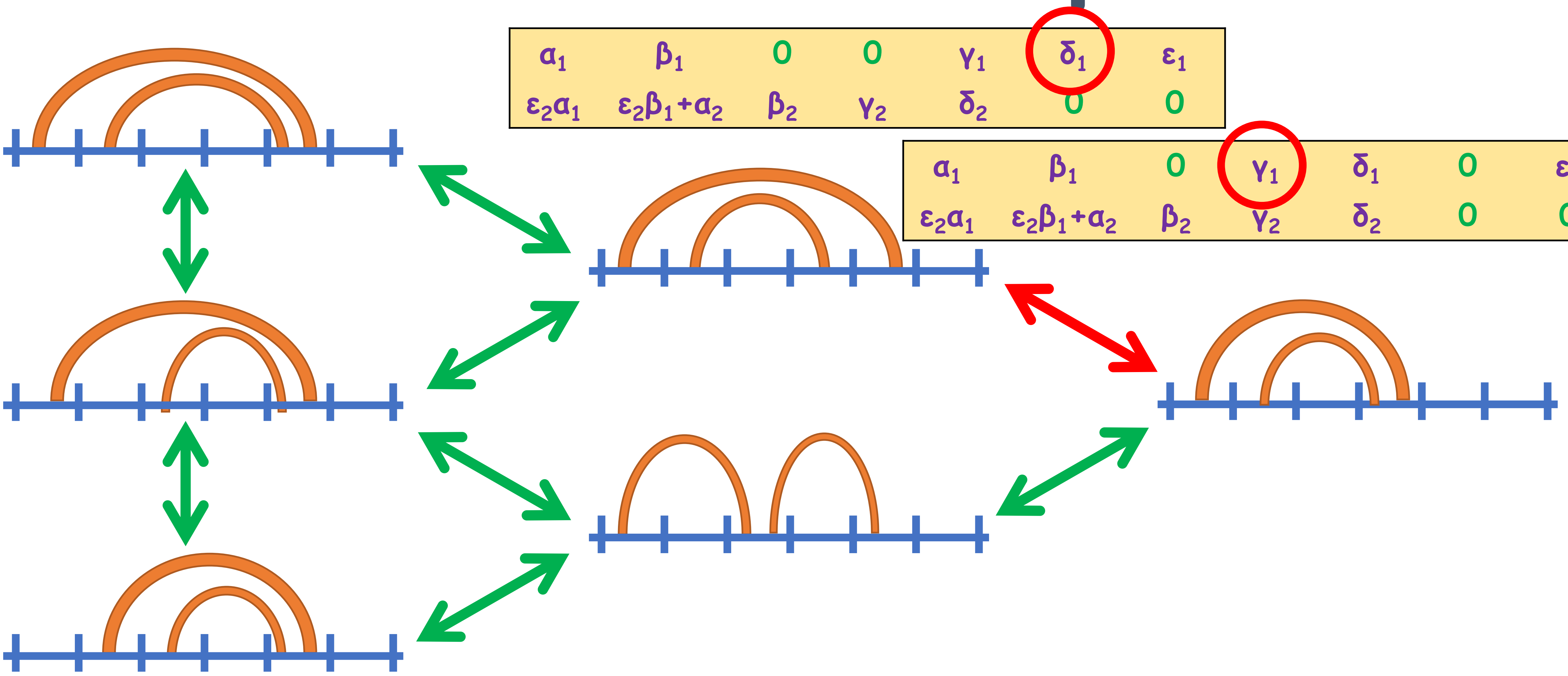
α_1	β_1	0	0	γ_1	δ_1	ε_1
$\varepsilon_2\alpha_1$	$\varepsilon_2\beta_1$	α_2	β_2	γ_2	δ_2	0

0	α_1	β_1	0	γ_1	δ_1	ε_1
0	$\varepsilon_2\alpha_1$	$\varepsilon_2\beta_1+\alpha_2$	β_2	γ_2	δ_2	0

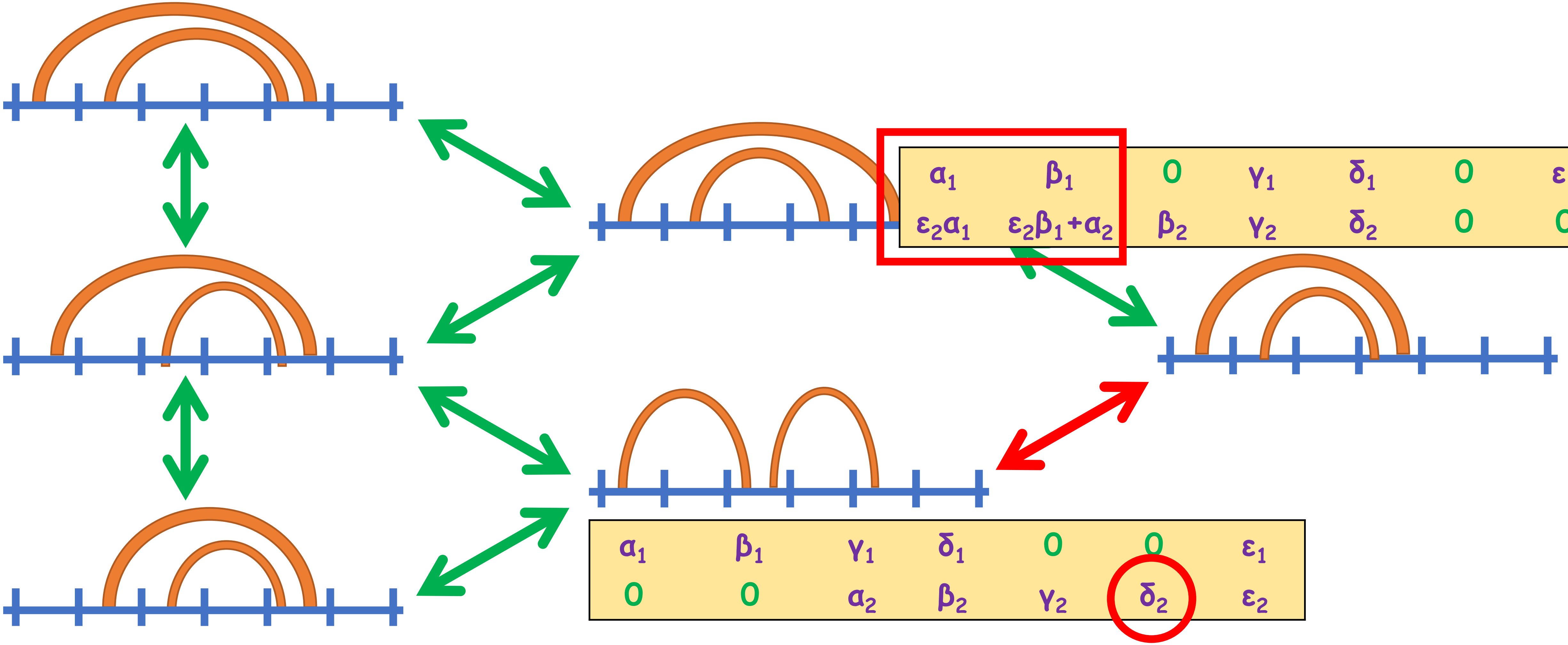
Boundaries Graph



Boundaries Graph



Boundaries Graph



THANK YOU