Triangular-Grid Billiards and Plabic Graphs

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Harvard

Based on joint work with Pakawut (Pro) Jiradilok.

FPSAC

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Triangular-Grid Billiards

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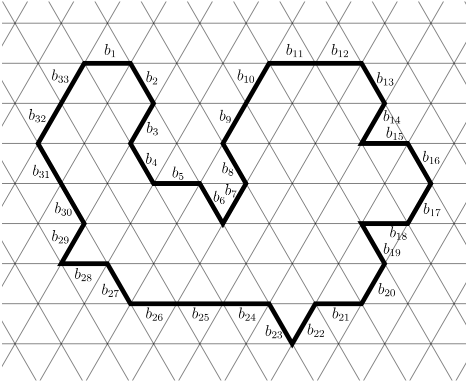

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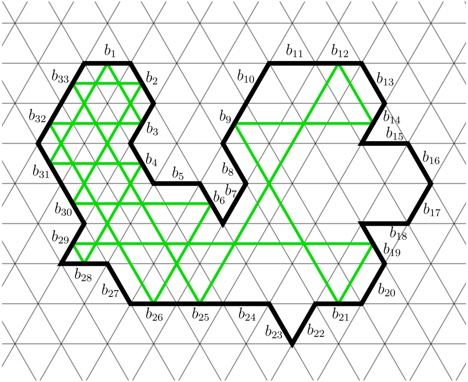

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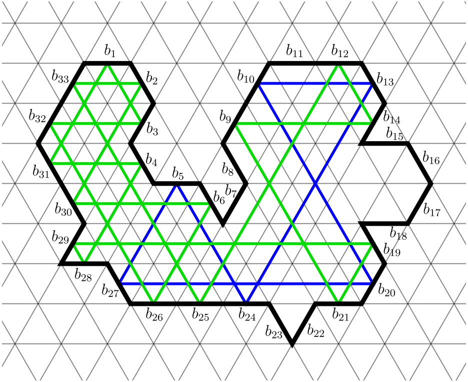

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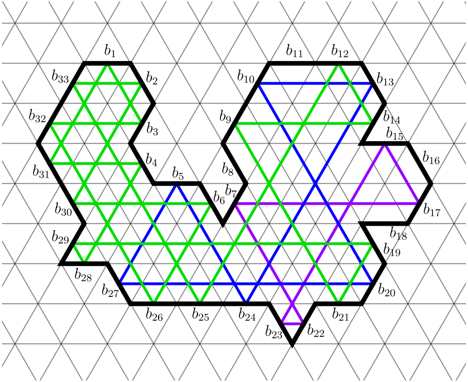

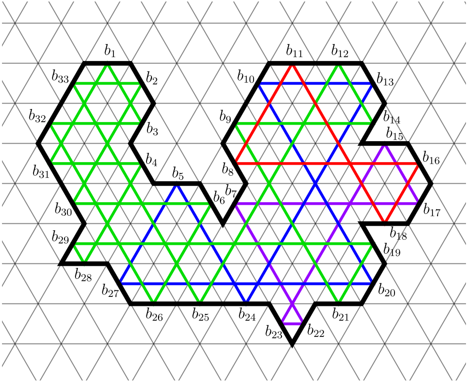
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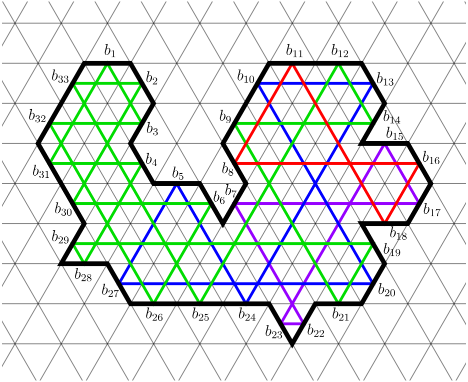

The *billiards permutation* π_P of this polygon P is

$$(1\,3\,32\,26\,6\,30\,2\,33\,25\,12\,14\,9\,21\,19\,29\,28\,4\,31)(5\,24\,13\,10\,20\,27)$$
$$(7\,22\,23\,15\,17)(8\,11\,18\,16).$$

Main Theorem

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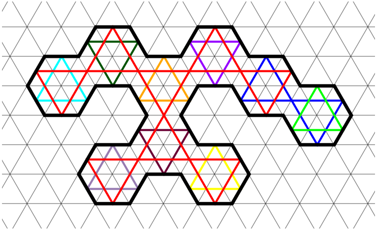
Throughout this talk, P is a polygon in the triangular grid.

Theorem (D.–Jiradilok, 2023)

We have $\operatorname{area}(P) \geq 6\operatorname{cyc}(P) - 6$ *and* $\operatorname{perim}(P) \geq \frac{7}{2}\operatorname{cyc}(P) - \frac{3}{2}$. *Also,* $\operatorname{area}(P) = 6\operatorname{cyc}(P) - 6$ *if and only if* P *is a "tree of unit hexagons."*



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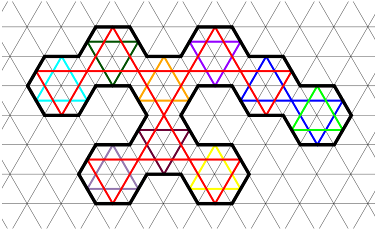

Conjecture (D.–Jiradilok, 2023)

We have $\mathrm{perim}(P) \geq 4\,\mathrm{cyc}(P) - 2$.

Breaking News

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Theorem (Honglin Zhu, last Friday++)

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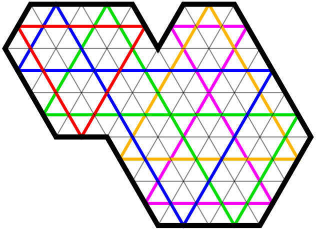

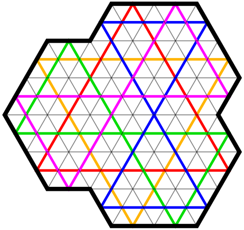

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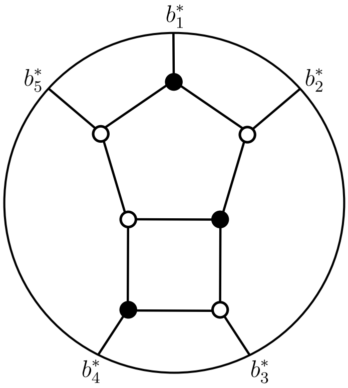

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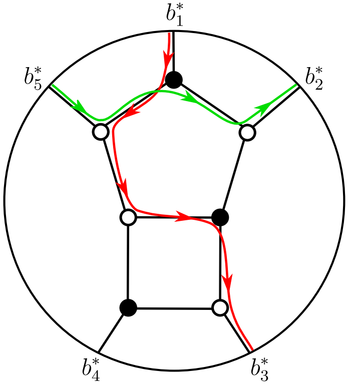

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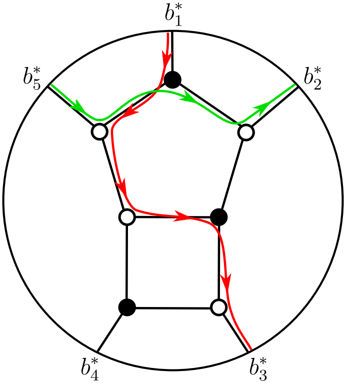
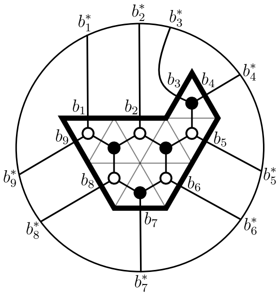
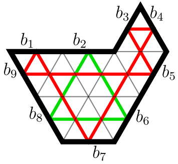
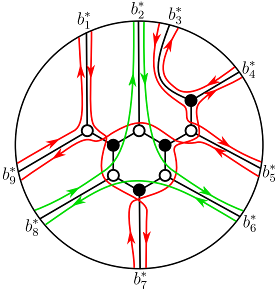


The *trip permutation* of this plabic graph is the cycle $(1\,3\,5\,2\,4)$.

Plabic Graphs from Grid Polygons







Essential Dimension

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Connected reduced plabic graphs with essential dimension 2 are exactly those coming from triangular-grid polygons.

Reformulation of Main Theorem

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Let G be a connected reduced plabic graph with essential dimension 2. Suppose G has n marked boundary points and v vertices, and let c be the number of cycles in the trip permutation π_G.

Corollary (D.–Jiradilok, 2023)

We have

$$v \geq 6c - 6 \quad \textit{and} \quad n \geq \frac{7}{2}c - \frac{3}{2}.$$

Corollary (Honglin Zhu, last Friday++)

We have

$$n \geq 4c - 2.$$

Future Directions: Other Families of Plabic Graphs

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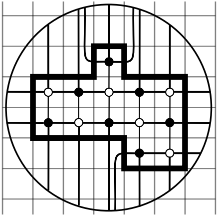
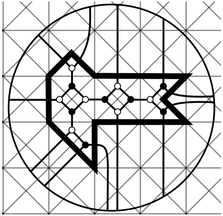
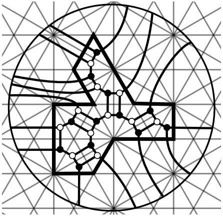
Let G be a connected reduced plabic graph with n marked boundary points, v vertices, and c cycles in its trip permutation.

Problem (D.–Jiradilok, 2023)

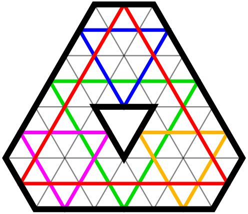
Find inequalities relating n and v to c when G is taken from some "interesting" family of plabic graphs.

Future Directions: Regions With Holes

Problem (D.–Jiradilok, 2023)

Obtain analogues of our results for billiards systems in triangular-grid polygons with holes cut out.



Future Directions: Random Polygons

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Question (D.–Jiradilok, 2023)

What can one say about $\text{cyc}(P)$ when P is a large random triangular-grid polygon?

Future Directions: Unitrajectorial Polygons

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Question (D.–Jiradilok, 2023)

What can one say about the triangular-grid polygons P such that $\operatorname{cyc}(P) = 1$?

Other Future Directions: Higher Dimensions

Fix a finite set P of alcoves of the affine Coxeter arrangement of type $\widetilde{A}_d$.

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$$\tau_j(w) = \begin{cases} s_j w & \text{if } s_j w \in P \\ w & \text{if } s_j w \notin P. \end{cases}$$

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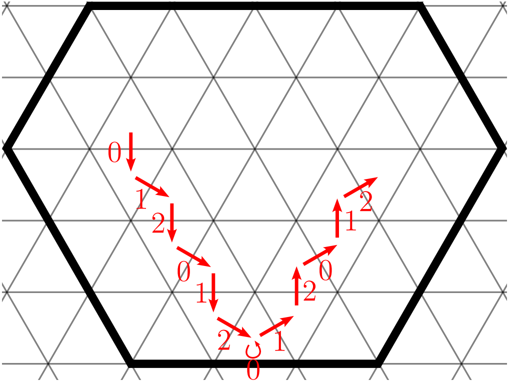
Start at an alcove in P and apply the sequence

$$\tau_0, \tau_1, \tau_2, \ldots, \tau_d, \tau_0, \tau_1, \tau_2, \ldots, \tau_d, \tau_0, \tau_1, \tau_2, \ldots, \tau_d, \ldots.$$

Other Future Directions: Higher Dimensions



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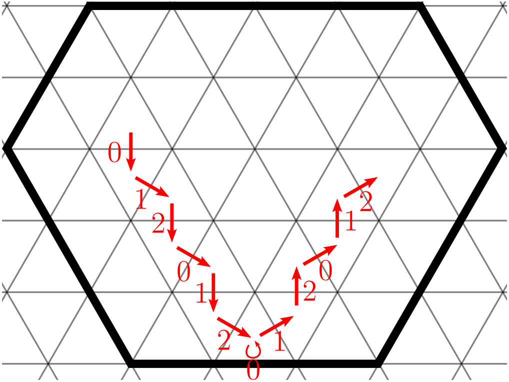
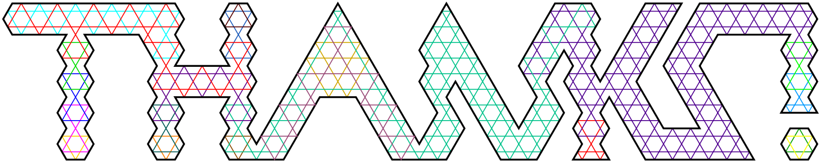


Problem (D.–Jiradilok, 2023)

Compare the number of trajectories in P with $|P|$.


THANKS!