Effective algebraicity for solutions of systems of functional equations with one catalytic variable

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Joint work with:
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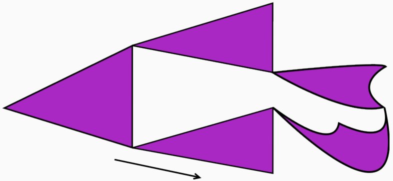
Motivation: Functional equations with one catalytic variable in combinatorics



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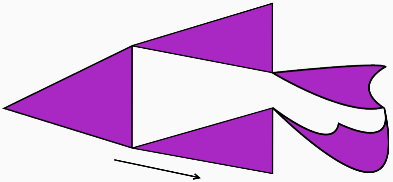

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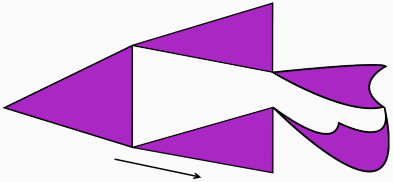

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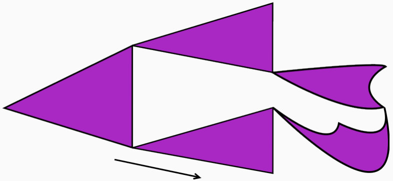

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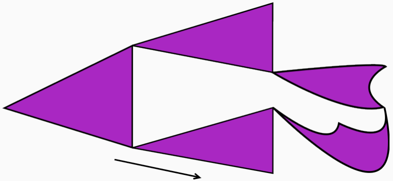

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$$81t^2 G(t)^3 - 9t(9t-2)G(t)^2 + (27t^2 - 66t + 1)G(t) - 3t^2 + 47t - 1 = 0$$

Previous works: case of a single equation

Theorem [Bousquet-Mélou, Jehanne '06] **see also [Popescu '86, Swan '98]**

Denote $\mathbb{K}$ a field of characteristic 0, denote $\Delta(F) := \frac{F(t,u)-\mathbf{F(t,1)}}{u-1}$ the discrete derivative operator. Let $f \in \mathbb{K}[u]$ and $Q \in \mathbb{K}[y_0, \dots, y_k, t, u]$ be polynomials. There exists a **unique** solution $F(t,u)$ in $\mathbb{K}[u][[t]]$ to

$$F(t,u) = f(u) + t \cdot Q(F(t,u), \Delta F(t,u), \dots, \Delta^{(k)} F(t,u), t, u). \tag{DDE}$$

Moreover, $F(t,u)$ is **algebraic** over $\mathbb{K}(t,u)$.

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- For **computing** the minimal polynomial of $\mathbf{F(t,1)}$, many contributors:
Banderier, Bender, Bostan, Bousquet-Mélou, Brown, Canfield, Chyzak, Flajolet, Gessel, Jehanne, N., Safey El Din, Tutte, Zeilberger, ...

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Goal: Classify the nature of $\mathbf{F(t,1)}$

Previous works (cont'd): case of systems of equations

- **Nature** of $\mathbf{F}(\mathbf{t},\mathbf{1})$: [Popescu '86] $\implies$ $\mathbf{F}(\mathbf{t},\mathbf{1})$ is **algebraic** over $\mathbb{K}(t)$

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- **Effective proof** + implicit algorithm: [Buchacher, Kauers '19] $\rightsquigarrow$ At FPSAC '19!

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Asinowski, Bacher, Banderier, Beaton, Bonichon, Bousquet-Mélou, Bouvel, Buchacher, Dorbec, Gittenberger, Guerrini, Jehanne, Kauers, Pennarun, Rinaldi, ...

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$\rightarrow$ **There should be things to say for any system of DDEs**

Main contribution

Theorem: [N., Yurkevich '23] **see also [Popescu '86, Swan '98]**

Let $n, k \geq 1$ be integers and $f_1, \dots, f_n \in \mathbb{K}[u]$, $Q_1, \dots, Q_n \in \mathbb{K}[y_1, \dots, y_{n(k+1)}, t, u]$ be polynomials. For $a \in \mathbb{K}$, set $\nabla^k F := F, \Delta F, \dots, \Delta^k F$. Then the **system** of DDEs

$$\begin{cases} (\mathbf{E_{F_1}}): \quad F_1 = f_1(u) + t \cdot Q_1(\nabla^k F_1, \dots, \nabla^k F_n, t, u), \\ \quad \vdots \qquad\qquad\qquad\qquad \vdots \\ (\mathbf{E_{F_n}}): \quad F_n = f_n(u) + t \cdot Q_n(\nabla^k F_1, \dots, \nabla^k F_n, t, u). \end{cases} \qquad \textbf{(SDDEs)}$$

admits a **unique** vector of solutions $(F_1, \dots, F_n) \in \mathbb{K}[u][[t]]^n$, and all its components are **algebraic** over $\mathbb{K}(t, u)$.

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Constructive proof $\Longrightarrow$ **Algorithm computing a polynomial annihilating $\mathbf{F_1(t, a)}$**

Finding more polynomial equations [Bonichon, Bousquet-Mélou, Dorbec, Pennarun '17]

Consider
$$\begin{cases} F(t,\mathbf{u}) = 1 + t\cdot\left(\mathbf{u} + 2\mathbf{u}F(t,\mathbf{u})^2 + 2\mathbf{u}G(t,1) + \mathbf{u}\frac{F(t,\mathbf{u})-\mathbf{u}\mathbf{F(t,1)}}{\mathbf{u}-1}\right), \\ G(t,\mathbf{u}) = t\cdot\left(2\mathbf{u}F(t,\mathbf{u})G(t,\mathbf{u}) + \mathbf{u}F(t,\mathbf{u}) + \mathbf{u}G(t,1) + \mathbf{u}\frac{G(t,\mathbf{u})-\mathbf{u}G(t,1)}{\mathbf{u}-1}\right). \end{cases}$$

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Multiplying by $\mathbf{u}-1$ gives $\rightsquigarrow F, G \equiv F(t,u), G(t,u)$

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Denote by $E_1, E_2 \in \mathbb{Q}(t)[x_1, x_2, \mathbf{z_0}, z_1, \mathbf{u}, t]$ polynomials such that

$$\text{for } i \in \{1,2\},\quad E_i(F(t,\mathbf{u}), G(t,\mathbf{u}), \mathbf{F(t,1)}, G(t,1), \mathbf{u}, t) = 0. \qquad (\equiv E_i(\mathbf{u}))$$

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Differentiating with respect to $\mathbf{u}$ yields

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For $\mathbf{U(t)} \in \bigcup_{d\geq 1} \overline{\mathbb{Q}}[[t^{\frac{1}{d}}]]$, $\begin{cases} \textbf{if} & (\partial_{x_1}E_1 \cdot \partial_{\mathbf{x_2}}E_2 - \partial_{x_1}E_2 \cdot \partial_{\mathbf{x_2}}E_1)(\mathbf{U(t)}) = 0, \\ \textbf{then} & (\partial_{x_1}E_1 \cdot \partial_{\mathbf{u}}E_2 - \partial_{x_1}E_2 \cdot \partial_{\mathbf{u}}E_1)(\mathbf{U(t)}) = 0. \end{cases}$

Sketch of strategy: differentiate with respect to the catalytic variable!

- **General case:** Denote the *numerator equations*

$$E_1(u) = 0, \dots, E_n(u) = 0, \text{ for } E_i \in \mathbb{K}[x_1, \dots, x_n, z_0, \dots, z_{nk-1}, \mathbf{u}, t].$$

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- Define:

$$\mathrm{Det} := \det \begin{pmatrix} \partial_{x_1} E_1 & \dots & \partial_{x_{n-1}} E_1 & \boldsymbol{\partial_{x_n} E_1} \\ \vdots & \ddots & \vdots & \\ \partial_{x_1} E_n & \dots & \partial_{x_{n-1}} E_n & \boldsymbol{\partial_{x_n} E_n} \end{pmatrix} \quad \text{and} \quad \mathrm{P} := \det \begin{pmatrix} \partial_{x_1} E_1 & \dots & \partial_{x_{n-1}} E_1 & \boldsymbol{\partial_{\mathbf{u}} E_1} \\ \vdots & \ddots & \vdots & \vdots \\ \partial_{x_1} E_n & \dots & \partial_{x_{n-1}} E_n & \boldsymbol{\partial_{\mathbf{u}} E_n} \end{pmatrix}$$

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- For $\mathbf{U(t)} \in \bigcup_{d \geq 1} \overline{\mathbb{K}}[[t^{\frac{1}{d}}]]$, $\mathrm{Det}(\mathbf{U(t)}) = 0$ imply $\mathrm{P}(\mathbf{U(t)}) = 0$

$\rightsquigarrow$ $\mathbf{n+2}$ **equations** and $\mathbf{n} + \mathbf{nk} + \mathbf{1}$ **unknowns**.

Sketch of strategy (cont'd): duplicate variables

Notations: $A(u) \equiv A(F_1(\mathbf{u}), \ldots, F_n(\mathbf{u}), F_1(a), \ldots, \partial_u^{k-1} F_n(a), t, \mathbf{u})$.

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- **If** $\mathrm{Det}(\mathbf{u}) = 0$ has *nk* **distinct solutions in u** in $\bigcup_{d \geq 1} \overline{\mathbb{K}}[[t^{\frac{1}{d}}]] \setminus \{a\}$, define

$$\mathcal{S}_{\mathrm{dup}} := \begin{cases} E_1(u_i) = E_2(u_i) = \cdots = E_n(u_i) = 0, \\ \mathrm{Det}(u_i) = 0, \mathrm{P}(u_i) = 0 \\ m \cdot \prod_{1 \leq \ell < j \leq nk} (u_\ell - u_j) - 1 = 0. \end{cases}, \text{ for } i = 1, \ldots, nk.$$

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- **Then:** $\mathbf{nk(n+2)+1}$ equations in $\mathbf{nk(n+2)+1}$ unknowns.

$$m, \underbrace{x_1, \ldots, x_{n^2k}}_{F_i(U_j)}, \underbrace{z_0, \ldots, z_{nk-1}}_{\partial^j F_i(t,a)}, \underbrace{u_1, \ldots, u_{nk}}_{U_i} \Rightarrow 1 + n^2k + nk + nk = \mathbf{nk(n+2)+1}.$$

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Hope: the **ideal** generated by $\mathcal{S}_{\mathrm{dup}}$ is **0-dimensional** over $\mathbb{K}(t)$.

A degenerate toy (SDDEs)

Modelling m-row restricted slicings: [Beaton, Bouvel, Guerrini, Rinaldi '19]

$$\begin{cases} F_1 = t^2\mathbf{u} + t^2\mathbf{u}\,(F_1 + F_2), \\ F_2 = t^2\mathbf{u}\,\frac{F_2 - F_2(t,1)}{\mathbf{u}-1} + t^2\mathbf{u}\,\frac{F_1 - F_1(t,1)}{\mathbf{u}-1} + t^2\mathbf{u}\,F_2, \end{cases}$$

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which gives

$$\begin{cases} E_1 := -x_1 + t^2\mathbf{u} + t^2\mathbf{u}(x_1 + x_2), \\ E_2 := -x_2(\mathbf{u} - 1) + t^2\mathbf{u}x_1 + t^2\mathbf{u}x_2 - t^2\mathbf{u}z_0 - t^2\mathbf{u}z_1. \end{cases}$$

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There exists **only one** solution $\mathbf{u} = \mathbf{U}(t) \in \bigcup_{d\geq 1} \overline{\mathbb{Q}}[[t^{\frac{1}{d}}]] \setminus \{1\}$ while $\mathbf{nk} = \mathbf{2}\ldots$

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First difficulty: How do we **create** more **relevant** solutions in $\mathbf{u}$ to $\mathbf{Det(u) = 0}$?

Symbolic deformation argument in the general case

Consider

$$\begin{cases} F_1 = f_1(\mathbf{u}) + t \cdot Q_1(\nabla^k F_1, \ldots, \nabla^k F_n, t, \mathbf{u}), \\ \quad \vdots \qquad\qquad\qquad \vdots \\ F_n = f_n(\mathbf{u}) + t \cdot Q_n(\nabla^k F_1, \ldots, \nabla^k F_n, t, \mathbf{u}). \end{cases} \tag{SDDEs}$$

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We define for $\gamma_{\mathbf{i},\mathbf{i}} = \mathbf{i}^{\mathbf{k}}$, $\gamma_{\mathbf{i},\mathbf{j}} = \mathbf{t}^{\beta}$ and for $\alpha, \beta \in \mathbb{N}$ such that $\alpha \gg \beta \gg 0$

$$\begin{cases} G_1 = f_1(\mathbf{u}) + t^{\alpha} \cdot Q_1(\nabla^k G_1, \nabla^k G_2, \ldots, \nabla^k G_n, t^{\alpha}, \mathbf{u}) + \boxed{\epsilon^k \cdot t \cdot \sum_{i=1}^{n} \gamma_{\mathbf{1},\mathbf{i}} \cdot \Delta^k G_i,} \\ \quad \vdots \qquad\qquad \vdots \\ G_n = f_n(\mathbf{u}) + t^{\alpha} \cdot Q_n(\nabla^k G_1, \nabla^k G_2, \ldots, \nabla^k G_n, t^{\alpha}, \mathbf{u}) + \boxed{\epsilon^k \cdot t \cdot \sum_{i=1}^{n} \gamma_{\mathbf{n},\mathbf{i}} \cdot \Delta^k G_i,} \end{cases} \in \mathbb{K}[\epsilon, u][[t]]$$

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- $\mathbf{G_i}(\mathbf{t}, \mathbf{u}, \epsilon)$ algebraic over $\mathbb{K}(t, \mathbf{u}, \epsilon) \Rightarrow \mathbf{G_i}(\mathbf{t}, \mathbf{u}, \mathbf{0}) = \mathbf{F_i}(\mathbf{t}^{\alpha}, \mathbf{u})$ algebraic over $\mathbb{K}(t, \mathbf{u})$.

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- $\mathbf{G_i}(\mathbf{t}, \mathbf{u}, \epsilon)$ **algebraic** over $\mathbb{K}(t, \mathbf{u}, \epsilon) \Rightarrow$ $\mathbf{G_i}(\mathbf{t}, \mathbf{u}, \mathbf{0}) = \mathbf{F_i}(\mathbf{t}^{\alpha}, \mathbf{u})$ **algebraic** over $\mathbb{K}(t, \mathbf{u})$.
- Denote the polynomial relations:

$$E_i(G_1, \dots, G_n, G_1(t, 0, \epsilon), \dots, \partial_u^{k-1} G_n(t, 0, \epsilon), t, \mathbf{u}, \epsilon) = 0, \text{ for } i = 1, \dots, n.$$

The deformation of (SDDEs) ensures good properties

Lemma 1: There are nk **distinct solutions** $\mathbf{u} = U_1, \ldots, U_{nk}$ in $\bigcup_{d \geq 1} \overline{\mathbb{K}}(\epsilon)[[t^{\frac{1}{d}}]]$ to $\mathrm{Det}(\mathbf{u}) = 0$.

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All in all: Effective Algebraic Geometry

Main steps for proving the invertibility of the Jacobian matrix

- Up to elementary operations, $\mathrm{Jac}_{\mathrm{dup}}(\mathcal{P})$ is an **upper-block triangular matrix**:

$$\mathrm{Jac}_{\mathrm{dup}}(\mathcal{P}) \sim \begin{pmatrix} \mathbf{A}(\mathbf{U_1}(\mathbf{t})) & 0 & \cdots & 0 & \star \\ 0 & \ddots & 0 & 0 & \star \\ 0 & \cdots & 0 & \mathbf{A}(\mathbf{U_{nk}}(\mathbf{t})) & \star \\ 0 & 0 & \cdots & 0 & \mathbf{\Lambda}(\mathbf{U_1}(\mathbf{t}), \dots, \mathbf{U_{nk}}(\mathbf{t})) \end{pmatrix},$$

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$$\mathrm{Jac}_{\mathrm{dup}}(\mathcal{P}) \sim \begin{pmatrix} \mathbf{A}(\mathbf{U_1}(\mathbf{t})) & 0 & \cdots & 0 & \star \\ 0 & \ddots & 0 & 0 & \star \\ 0 & \cdots & 0 & \mathbf{A}(\mathbf{U_{nk}}(\mathbf{t})) & \star \\ 0 & 0 & \cdots & 0 & \mathbf{\Lambda}(\mathbf{U_1}(\mathbf{t}), \dots, \mathbf{U_{nk}}(\mathbf{t})) \end{pmatrix},$$

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- For all $1 \leq i \leq nk$, we have $\mathbf{val_t}(\mathbf{det}(\mathbf{A}(\mathbf{U_i}(\mathbf{t})))) < +\infty$,
- Finally, there exists $\gamma \in \mathbb{N}$ such that

$$\det(\mathbf{\Lambda}(\mathbf{U_1}(\mathbf{t}), \ldots, \mathbf{U_{nk}}(\mathbf{t}))) = \prod_{i} U_i(t)^{\gamma} \cdot \prod_{i<j} (U_i(t) - U_j(t)) \cdot \mathbf{H}(\mathbf{t}) \mod t^{\alpha},$$

for some $\mathbf{H}(\mathbf{t}) \in \mathbb{K}[t, \epsilon] \setminus \{0\}$ whose degree is independent of α.

Summary of our new method

Starting from

$$\begin{cases} F_1 = f_1(u) + t \cdot Q_1(\nabla^k F_1, \ldots, \nabla^k F_n, t, u), \\ \quad \vdots \qquad\qquad\qquad \vdots \\ F_n = f_n(u) + t \cdot Q_n(\nabla^k F_1, \ldots, \nabla^k F_n, t, u). \end{cases} \qquad \textbf{(SDDEs)}$$

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3. **Compute** a non-trivial element of $(\langle \mathcal{S}_{\text{dup}} \rangle : \det(\mathrm{Jac}_{\mathcal{S}_{\text{dup}}})^\infty) \cap \mathbb{K}[t, z_0, \epsilon]$, then set ϵ to 0.

Conclusion and perspectives

- Solutions of systems of DDEs are algebraic series,
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Effective algebraicity for solutions of systems of functional equations with one catalytic variable

Hadrien Notarantonio[1,2], Sergey Yurkevich[1,3]

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(Up coming)

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(i) Detailed algebraicity proof,

(ii) New efficient algorithms avoiding the duplication of variables.

On Solutions of Systems of Discrete Differential Equations

Hadrien Notarantonio[1] and Sergey Yurkevich[2]

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Abstract

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Thanks for your attention!

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