Bruhat interval polytopes, 1-skeleton lattices, and smooth torus orbit closures

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Bruhat order

The **Bruhat order** $\preceq$ is a partial order on the symmetric group S_n with cover relations $w \prec w \cdot (ij)$ whenever

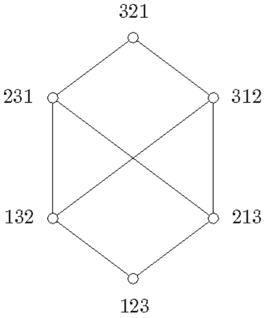
$$\ell(w(ij)) = \ell(w) + 1.$$

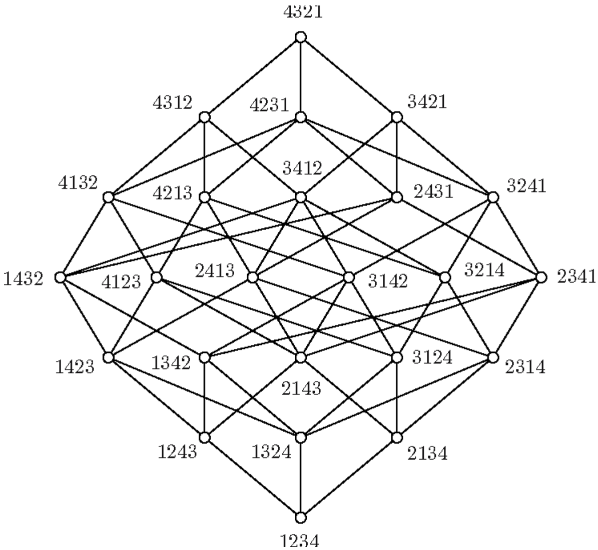
Here $\ell(w)$ is the number of inversions of w, or equivalently the smallest ℓ such that

$$w = s_{i_1} \cdots s_{i_\ell},$$

where $s_i = (i\ i+1)$ is an adjacent transposition.

Bruhat order on S_3 and S_4





Bruhat interval polytopes

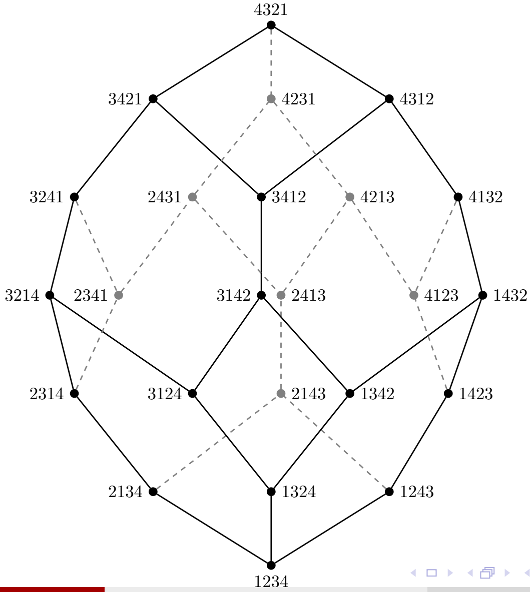
Definition (Kodama–Williams)

For $v \in S_n$, the **Bruhat interval polytope** Q_v is the polytope in $\mathbb{R}^n$ whose vertices are the permutations $u \preceq v$ (viewing permutations in one-line notation as vectors).

Example

If $v = w_0$ then $[e, v] = S_n$ is the whole symmetric group. In this case Q_v is the **permutohedron**.

The permutohedron



Why Bruhat interval polytopes?

The Bruhat interval polytope Q_v is:

- The *moment map* image of a generic torus orbit closure Y_v in Schubert variety X_v;
- The moment map image of the totally nonnegative part of X_v;
- A *Coxeter matroid polytope*;
- Isomorphic to a *Bridge polytope* when and v is Grassmannian.

Properties of Bruhat interval polytopes

Theorem (Tsukerman and Williams 2015)

Any edge of Q_v is a Bruhat cover relation; in particular, Q_v is a **generalized permutohedron**.

- This means that the normal fan of Q_v coarsens the **braid fan**: the fan determined by the hyperplane arrangement $\{x_i - x_j = 0 \mid 1 \leq i < j \leq n\}$.
- Thus the top-dimensional cones of the normal fan induce an equivalence relation Θ_v on S_n.

The 1-skeleton poset

We will be interested in the following poset:

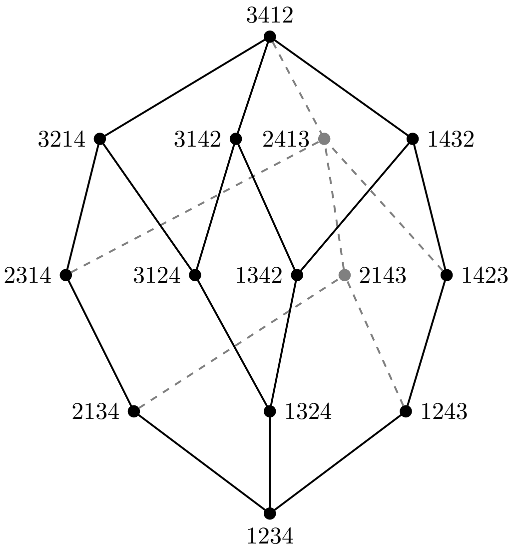
Definition

Let P_v be the poset on $[e, v]$ with cover relations $x \lessdot_v y$ if $\overline{xy}$ is an edge of Q_v and $\ell(y) = \ell(x) + 1$.

Example

If $v = w_0$, then P_v is **weak order** on S_n.

Example: $v = 3412$



P_v is a quotient

Proposition

Let Θ_v be the equivalence relation on S_n induced by the normal fan of Q_v, then

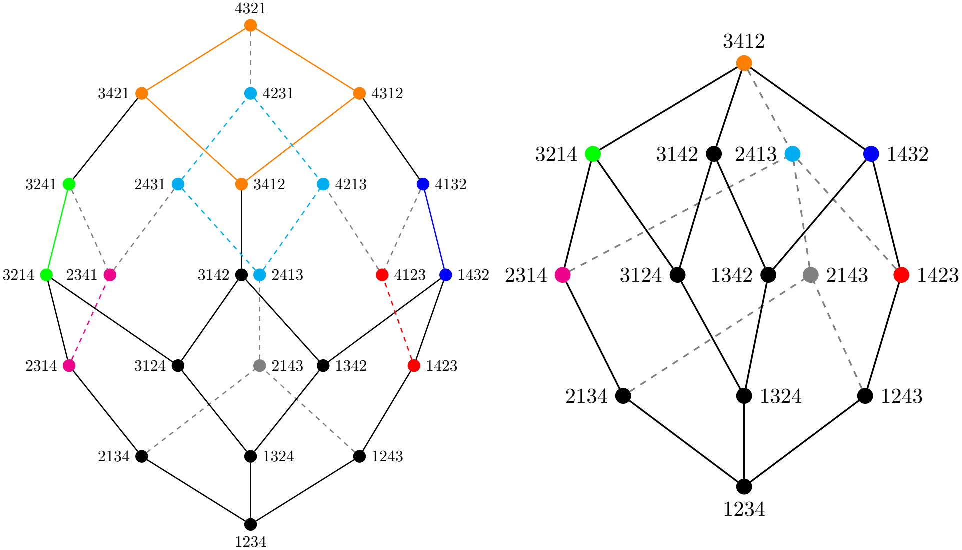
$$P_v \cong P_{w_0}/\Theta$$

as posets.

Each equivalence class $[x]_\Theta$ contains a unique element from $[e, v]$.

The map $x \mapsto [e, v] \cap [x]_\Theta$ is the *matroid map*.

Example: $v = 3412$



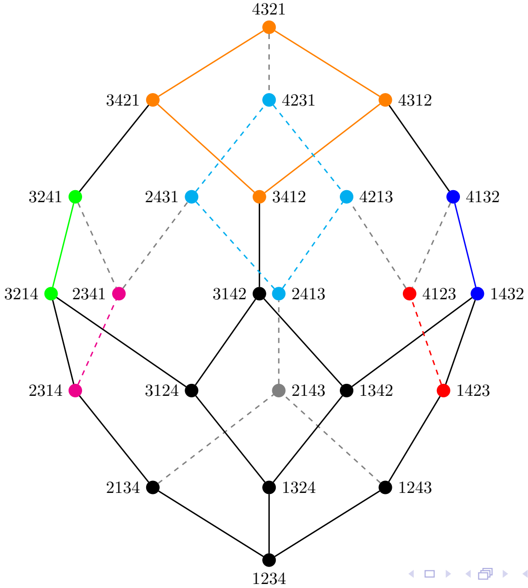
The top and bottom maps

Theorem

Each equivalence class $[x]_\Theta$ *contains a unique minimal element* $\mathrm{bot}(x)$ *and unique maximal element* $\mathrm{top}(x)$ *under weak order.*

Note: The existence of $\mathrm{bot}(x)$ is straightforward, this is just the matroid map. The existence of $\mathrm{top}(x)$ is much more surprising.

Example of top and bottom maps



The lattice property

Proposition (Well-known)

The weak order is a **lattice**.

Theorem

The map top : $S_n \to S_n$ *preserves weak order (unlike* bot*!). This determines the join operation on* P_v*:*

$$x \vee_{P_v} y = \text{bot}\left(\text{top}(x) \vee_{weak} \text{top}(y)\right).$$

Corollary

The poset P_v *is a lattice.*

Note: The *meet* operation does not come from $\wedge_{weak}$. So P_v is a join-semilattice quotient but not a lattice quotient of weak order.

Special cases

Example

The atoms of P_v are just the simple reflections s_i which are in the support of v. For any set J of simple reflections, we have

$$\bigvee_{s \in J} s = m(v, J)$$

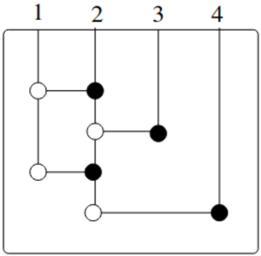
computes the *parabolic map* of Billey–Fan–Losonczy, the longest element of W_J lying in $[e, v]$.

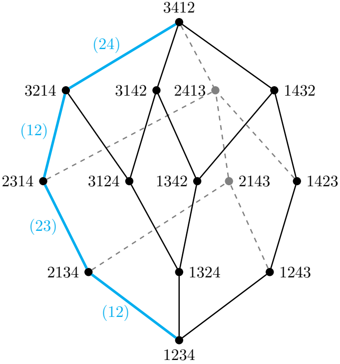
Example

When v is Grassmannian, Q_v is isomorphic to a *Bridge polytope*. It was conjectured by Fraser that P_v is a lattice, in this case.

Bridge polytopes and BCFW-bridge decompositions

BCFW bridge decompositions: Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka
Correspondence to paths in Bridge polytope: Williams

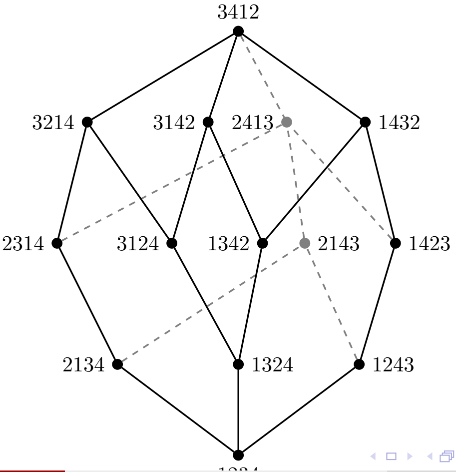




Vertex-degree monotonicity

Theorem

If $x \leq_{P_v} y$ then $\deg(x) \leq \deg(y)$ as vertices in Q_v.



Simple BIPs

Recall that the *Schubert variety* $X_v \subset G/B$ is $\overline{BvB/B}$. Write Y_v for a generic T-orbit closure in X_v, this is a toric variety with associated polytope Q_v.

Corollary (Conjectured by Lee–Masuda)

The polytope Q_v is simple if and only if it is simple at the vertex v. Equivalently, the variety Y_v is smooth if and only if it is smooth at vB.

Directionally simple polytopes

Let $G_c(Q)$ denote the directed graph on the 1-skeleton of a polytope Q according to increasing inner product with the vector c.

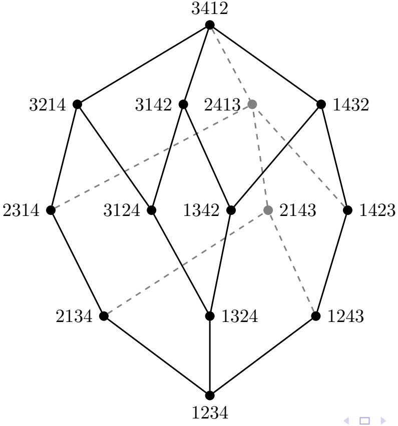
Definition

We say that a polytope $Q \subset \mathbb{R}^d$ is *directionally simple* with respect to the generic cost vector c if for every vertex v of Q and every set E of edges of $G_c(P)$ with source v there exists a face F of Q containing v whose set of edges incident to v is exactly E.

BIPs are directionally simple

Theorem (Different proof by Lee–Masuda–Park)

The polytope Q_v is directionally simple for all $v \in S_n$ when edges are oriented toward elements of higher length.



f-vectors and h-vectors

For a polytope $Q \subset \mathbb{R}^d$, write $(f_0, f_1, \dots, f_d)$ for the number of vertices, edges,. . . . This is the f-**vector**.

The h-**vector** $h(Q) = (h_0, \dots, h_d)$ is defined by the equality of polynomials

$$\sum_{i=0}^{d} f_i(x-1)^i = \sum_{k=0}^{d} h_k x^k.$$

Theorem (Dehn–Somerville equations)

If Q is simple, then $h(Q)$ is positive, symmetric, and unimodal.

The h-vector of Q_v

The ***h*-vector** $h(Q) = (h_0, \dots, h_d)$ is defined by the equality of polynomials

$$\sum_{i=0}^{d} f_i(x-1)^i = \sum_{k=0}^{d} h_k x^k.$$

Theorem

For any $v \in S_n$, the h-vector $h(Q_v)$ is given by

$$h_i = |\{x \in [e, v] \textit{ such that } \operatorname{des}(\operatorname{top}(x)) = n - i - 1\}|.$$

In particular, $h_i \geq 0$.

Thanks for listening!