Symmetric group characters are computationally hard

$\chi_{\lambda}(\alpha)^2$ or $|\chi_{\lambda}(\alpha)|$ do NOT have a nice combinatorial interpretation!

Greta Panova
joint work with Christian Ikenmeyer and Igor Pak

University of Southern California

FPSAC 2023
Symmetric group representations

$S_n$ – permutations under composition:

$$\pi : [1, 2, \ldots, n] \simto [1, 2, \ldots, n], \quad \pi \sigma = \pi(\sigma)$$
Symmetric group representations

$S_n$ – permutations under composition:

\[ \pi : [1, 2, \ldots, n] \xrightarrow{\sim} [1, 2, \ldots, n], \quad \pi \sigma = \pi(\sigma) \]

Representations: homomorphism $S_n \to GL_N(\mathbb{C})$

Example: if $V = \mathbb{C}^3$, $\pi \in S_3$, set $\pi(e_i) := e_{\pi(i)}$ for $i = 1..3$, so e.g. $231 \to \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$
**Symmetric group representations**

$S_n$ – permutations under composition:

$$\pi : [1, 2, \ldots, n] \sim [1, 2, \ldots, n], \quad \pi \sigma = \pi(\sigma)$$

Representations: homomorphism $S_n \to GL_N(\mathbb{C})$

Example: if $V = \mathbb{C}^3$, $\pi \in S_3$, set $\pi(e_i) := e_{\pi(i)}$ for $i = 1..3$, so e.g. $231 \to \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

The **irreducible representations** of the **symmetric group** $S_n$: the **Specht modules** $S_\lambda$

$$V = \mathbb{C}(e_1 + e_2 + e_3) \oplus \mathbb{C}(e_1 - e_2, e_2 - e_3)$$

$S_{(3)} \simeq V_1 \quad S_{(2,1)} \simeq V_2$

Basis indexed by SYTs of shape $\lambda$, so $\dim S_\lambda = f^\lambda := \# \{ T : \text{SYT}, \text{shape } \lambda \}$.

$$\begin{array}{cccccc}
1 & 2 & 1 & 2 & 1 & 3 \\
3 & 4 & 3 & 5 & 2 & 4 \\
5 & & & & 2 & 5 \\
3 & 4 & 5 & & & 3 \\
& 2 & 5 & & 4 & \\
& & 2 & 5 & & \\
\end{array}$$
Symmetric group representations

$S_n$ – permutations under composition:

$$\pi : [1, 2, \ldots, n] \sim [1, 2, \ldots, n], \quad \pi \sigma = \pi(\sigma)$$

Representations: homomorphism $S_n \to GL_N(\mathbb{C})$

Example: if $V = \mathbb{C}^3$, $\pi \in S_3$, set $\pi(e_i) := e_{\pi i}$ for $i = 1..3$, so e.g. $231 \to \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

The irreducible representations of the symmetric group $S_n$: the Specht modules $S_\lambda$

$$V = \mathbb{C}\langle e_1 + e_2 + e_3 \rangle \oplus \mathbb{C}\langle e_1 - e_2, e_2 - e_3 \rangle$$

$S(3) \simeq V_1 \quad S(2,1) \simeq V_2$

Basis indexed by SYTs of shape $\lambda$, so dim $S_\lambda = f_\lambda := \# \{ T : \text{SYT, shape } \lambda \}$.

123 125 134 135 145 234 235 245 345

Characters: $\text{char } S_\lambda = \chi^\lambda : S_n \to \mathbb{C}$

$$\chi^{(2,1)}(231) = \text{Trace} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = -1$$
Structure constants

Tensor product of irreducible $GL$ representations (Weyl modules $V_\alpha$):

$$V_\lambda \otimes V_\mu = \bigoplus_\nu V_\nu^{\oplus c^\nu_{\lambda \mu}}$$

Littlewood-Richardson coefficients:

$$c^\nu_{\lambda \mu} = \langle \chi^{\lambda} \times \chi^{\mu}, \chi^{\nu} \downarrow_{S_k \times S_{n-k}}^S \rangle$$
Structure constants

Tensor product of irreducible $GL$ representations (Weyl modules $V_\alpha$):

$$V_\lambda \otimes V_\mu = \bigoplus_\nu V_\nu^{c^\nu_{\lambda\mu}}$$

Littlewood-Richardson coefficients: $c^\nu_{\lambda\mu} = \langle \chi^{\lambda} \times \chi^{\mu}, \chi^{\nu} \downarrow_{S_k \times S_{n-k}}^S \rangle$

Theorem (Littlewood-Richardson, stated 1934, proven 1970's)

*The coefficient $c^\nu_{\lambda\mu}$ is equal to the number of LR tableaux of shape $\nu/\mu$ and type $\lambda$.*

(LR tableaux of shape $(6, 4, 3)/(3, 1)$ and type $(4, 3, 2)$. $c^{(6,4,3)}_{(3,1)(4,3,2)} = 2$)
**Structure constants**

Tensor product of irreducible GL representations (Weyl modules $V_\alpha$):

$$V_\lambda \otimes V_\mu = \bigoplus \nu V_\nu^{c^\nu_{\lambda \mu}}$$

**Littlewood-Richardson coefficients:** $c^\nu_{\lambda \mu} = \langle \chi^\lambda \times \chi^\mu, \chi^\nu \downarrow^{S_n}_{S_k \times S_{n-k}} \rangle$

**Theorem (Littlewood-Richardson, stated 1934, proven 1970’s)**

The coefficient $c^\nu_{\lambda \mu}$ is equal to the number of LR tableaux of shape $\nu/\mu$ and type $\lambda$.

(LR tableaux of shape $(6, 4, 3)/(3, 1)$ and type $(4, 3, 2)$. $c^{(6,4,3)}_{(3,1)(4,3,2)} = 2$)

**Kronecker coefficients:** $g(\lambda, \mu, \nu)$ – multiplicity of $S_\nu$ in $S_\lambda \otimes S_\mu$

$$S_\lambda \otimes S_\mu = \bigoplus \nu \vdash n S_\nu^{g(\lambda, \mu, \nu)}$$

$$g(\lambda, \mu, \nu) = \langle \chi^\lambda \chi^\mu, \chi^\nu \rangle = \frac{1}{n!} \sum_{w \in S_n} \chi^\lambda(w)\chi^\mu(w)\chi^\nu(w)$$
Structure constants

Tensor product of irreducible $GL$ representations (Weyl modules $V_\alpha$):

$$V_\lambda \otimes V_\mu = \bigoplus \nu V_\nu^{c_{\lambda\mu}^\nu}$$

Littlewood-Richardson coefficients: $c_{\lambda\mu}^\nu = \langle \chi^\lambda \times \chi^\mu, \chi^\nu \downarrow_{S_k \times S_{n-k}} \rangle$

Theorem (Littlewood-Richardson, stated 1934, proven 1970’s)

The coefficient $c_{\lambda\mu}^\nu$ is equal to the number of LR tableaux of shape $\nu/\mu$ and type $\lambda$.

(LR tableaux of shape $(6, 4, 3)/(3, 1)$ and type $(4, 3, 2)$. $c_{(3,1)(4,3,2)}^{(6,4,3)} = 2$)

Kronecker coefficients: $g(\lambda, \mu, \nu)$ – multiplicity of $S_\nu$ in $S_\lambda \otimes S_\mu$

$$S_\lambda \otimes S_\mu = \bigoplus_{\nu \vdash n} S_\nu^{g(\lambda, \mu, \nu)}$$

$$g(\lambda, \mu, \nu) = \langle \chi^\lambda \chi^\mu, \chi^\nu \rangle = \frac{1}{n!} \sum_{w \in S_n} \chi^\lambda(w) \chi^\mu(w) \chi^\nu(w)$$

Plethysm coefficients: $GL_n \xrightarrow{\rho^\nu} GL_m \xrightarrow{\rho^\mu} GL_N$: $\rho^\mu \circ \rho^\nu : GL_n \rightarrow GL_N$:

$$\rho^\mu(\rho^\nu) = \bigoplus_{\lambda} V_\lambda^{a_{\lambda}(\mu[\nu])}$$
Major problems in Algebraic Combinatorics

[Murnaghan, 1938]: \[ c_{\mu \nu}^\lambda = g \left( (N - |\lambda|, \lambda), (N - |\mu|, \mu), (N - |\nu|, \nu) \right) \] for \(|\lambda| = |\mu| + |\nu|\) and \(N\)-large.
Major problems in Algebraic Combinatorics

[Murnaghan, 1938]: \( c_{\mu,\nu}^\lambda = g \left( (N - |\lambda|, \lambda), (N - |\mu|, \mu), (N - |\nu|, \nu) \right) \) for \(|\lambda| = |\mu| + |\nu|\) and \(N\)-large.

Problem (Murnaghan 1938.. Lascoux, Garsia-Remmel 1980s... Stanley 2000)

*Find a positive combinatorial interpretation for \( g(\lambda, \mu, \nu) \), i.e. a family of combinatorial objects \( \mathcal{O}_{\lambda, \mu, \nu} \), s.t. \( g(\lambda, \mu, \nu) = \# \mathcal{O}_{\lambda, \mu, \nu} \).*
Major problems in Algebraic Combinatorics

[Murnaghan, 1938]: \[ c^{\lambda}_{\mu\nu} = g \left( (N - |\lambda|, \lambda), (N - |\mu|, \mu), (N - |\nu|, \nu) \right) \] for \(|\lambda| = |\mu| + |\nu|\) and \(N\)-large.

Problem (Murnaghan 1938.. Lascoux, Garsia-Remmel 1980s... Stanley 2000)

*Find a positive combinatorial interpretation for \(g(\lambda, \mu, \nu)\), i.e. a family of combinatorial objects \(O_{\lambda, \mu, \nu}\), s.t. \(g(\lambda, \mu, \nu) = \#O_{\lambda, \mu, \nu}\).*

Combinatorial formulas for \(g(\lambda, \mu, \nu)\):

- Two two-row partitions [Remmel–Whitehead, 1994; Blasiak–Mulmuley–Sohoni, 2015];
- One two-row and other restrictions [Ballantine-Orellana, 2006]
- One hook \(\nu = (n - k, 1^k)\) [Blasiak 2012, Blasiak-Liu 2014]
- Other special cases [Bessenrodt-Bowman, Colmenarejo-Rosas, Ikenmeyer-Mulmuley-Walter, Pak-Panova, Tewari, Vallejo].
Major problems in Algebraic Combinatorics

[Murnaghan, 1938]: \[ c_{\mu \nu}^\lambda = g \left( \left( N - |\lambda|, \lambda \right), \left( N - |\mu|, \mu \right), \left( N - |\nu|, \nu \right) \right) \] for \(|\lambda| = |\mu| + |\nu|\) and \(N\)-large.

Problem (Murnaghan 1938.. Lascoux, Garsia-Remmel 1980s... Stanley 2000)

*Find a positive combinatorial interpretation for \(g(\lambda, \mu, \nu)\), i.e. a family of combinatorial objects \(O_{\lambda, \mu, \nu}\), s.t. \(g(\lambda, \mu, \nu) = \#O_{\lambda, \mu, \nu}\).*

Combinatorial formulas for \(g(\lambda, \mu, \nu)\):

- Two two-row partitions [Remmel–Whitehead, 1994; Blasiak–Mulmuley–Sohoni, 2015];
- One two-row and other restrictions [Ballantine-Orellana, 2006]
- One hook \(\nu = (n - k, 1^k)\) [Blasiak 2012, Blasiak-Liu 2014]
- Other special cases [Bessenrodt-Bowman, Colmenarejo-Rosas, Ikenmeyer-Mulmuley-Walter, Pak-Panova, Tewari, Vallejo].

Problem (Stanley 2000)

*Find a positive combinatorial interpretation for \(a_\lambda(d[n])\).*
Major problems in Algebraic Combinatorics

[Murnaghan, 1938]: $c_{\mu \nu}^{\lambda} = g \left( (N - |\lambda|, \lambda), (N - |\mu|, \mu), (N - |\nu|, \nu) \right)$ for $|\lambda| = |\mu| + |\nu|$ and $N$-large.

Problem (Murnaghan 1938.. Lascoux, Garsia-Remmel 1980s... Stanley 2000)

Find a positive combinatorial interpretation for $g(\lambda, \mu, \nu)$, i.e. a family of combinatorial objects $O_{\lambda, \mu, \nu}$, s.t. $g(\lambda, \mu, \nu) = \#O_{\lambda, \mu, \nu}$.

Combinatorial formulas for $g(\lambda, \mu, \nu)$:

- Two two-row partitions [Remmel–Whitehead, 1994; Blasiak–Mulumley–Sohoni, 2015];
- One two-row and other restrictions [Ballantine-Orellana, 2006]
- One hook $\nu = (n - k, 1^k)$ [Blasiak 2012, Blasiak-Liu 2014]
- Other special cases [Bessenrodt-Bowman, Colmenarejo-Rosas, Ikenmeyer-Mulumley-Walter, Pak-Panova, Tewari, Vallejo].

Problem (Stanley 2000)

Find a positive combinatorial interpretation for $a_{\lambda}(d[n])$.

Applications beyond Combinatorics: Geometric Complexity Theory (VP vs VNP...)
Major problems in Algebraic Combinatorics

[Murnaghan, 1938]: \( c_{\mu\nu}^{\lambda} = g \left( (N - |\lambda|, \lambda), (N - |\mu|, \mu), (N - |\nu|, \nu) \right) \) for \(|\lambda| = |\mu| + |\nu|\) and \(N\)-large.

Problem (Murnaghan 1938.. Lascoux, Garsia-Remmel 1980s... Stanley 2000)

Find a positive combinatorial interpretation for \( g(\lambda, \mu, \nu) \), i.e. a family of combinatorial objects \( O_{\lambda,\mu,\nu} \), s.t. \( g(\lambda, \mu, \nu) = \#O_{\lambda,\mu,\nu} \).

Combinatorial formulas for \( g(\lambda, \mu, \nu) \):

- Two two-row partitions [Remmel–Whitehead, 1994; Blasiak–Mumuley–Sohoni, 2015];
- One two-row and other restrictions [Ballantine–Orellana, 2006]
- One hook \( \nu = (n - k, 1^k) \) [Blasiak 2012, Blasiak-Liu 2014]
- Other special cases [Bessenrodt-Bowman, Colmenarejo-Rosas, Ikenmeyer-Mumuley-Walter, Pak-Panova, Tewari, Vallejo].

Problem (Stanley 2000)

Find a positive combinatorial interpretation for \( a_{\lambda}(d[n]) \).

Applications beyond Combinatorics: Geometric Complexity Theory (VP vs VNP...)

What is really a “combinatorial interpretation”? → NEXT
**Computational Complexity**

**Decision problems:**

- **Input**: $I$
- **Algorithm**: is $I \in L$?
- **Yes/No**

Ex: $L = Primes$, $I$ – an integer
Computational Complexity

Decision problems: Input $I$ → Algorithm: is $I \in L$? → Yes/No

Ex: $L = \text{Primes}$, $I$ – an integer
Input size($I$) = $n$ bits
Computational Complexity

Decision problems: Input $l$ $\rightarrow$ Algorithm: is $l \in L$? $\rightarrow$ Yes/No

Ex: $L = Primes$, $l$ – an integer
Input $size(l) = n$ bits

$P = \text{yes/no answer in time } O(n^k)$ some fixed $k$. 
Computational Complexity

Decision problems: Input $i$ $\rightarrow$ Algorithm: is $i \in L$? $\rightarrow$ Yes/No

Ex: $L = Primes$, $i$ – an integer
Input size($I$) = $n$ bits

$P$ = yes/no answer in time $O(n^k)$ some fixed $k$.
Ex: PRIMES, Linear Programming, Graph connectivity ...
Computational Complexity

Decision problems: Input $I$ $\rightarrow$ Algorithm: is $I \in L?$ $\rightarrow$ Yes/No

Ex: $L = Primes$, $I$ – an integer
Input size($I$) = $n$ bits
$\mathbf{P} = \text{yes/no answer in time } O(n^k) \text{ some fixed } k.$
Ex: PRIMES, Linear Programming, Graph connectivity ...

$\mathbf{NP} = \text{“yes” can be verified in } O(n^k) \text{ for some fixed } k, \text{ i.e. there is a “poly-time witness”}.$
Decision problems: Input \( I \) → Algorithm: is \( I \in L \) → Yes/No

Ex: \( L = \text{Primes} \), \( I \) – an integer
Input size(\( I \)) = \( n \) bits

\( \mathbf{P} \) = yes/no answer in time \( O(n^k) \) some fixed \( k \).

Ex: PRIMES, Linear Programming, Graph connectivity ...

\( \mathbf{NP} \) = “yes” can be verified in \( O(n^k) \) for some fixed \( k \), i.e. there is a “poly-time witness”.
Ex: Input \( I = G \) - graph, \( L \) = graphs with Hamiltonian cycles, the witness is the cycle \( v_1 - \ldots - v_m - v_1 \)
Computational Complexity

**Decision problems:**

Input $I$ $\rightarrow$ Algorithm: is $I \in L$? $\rightarrow$ Yes/No

Ex: $L = \text{Primes}$, $I$ – an integer
Input size($I$) = $n$ bits

$\mathbf{P}$ = yes/no answer in time $O(n^k)$ some fixed $k$.
Ex: PRIMES, Linear Programming, Graph connectivity ...

$\mathbf{NP}$ = “yes” can be verified in $O(n^k)$ for some fixed $k$, i.e. there is a “poly-time witness”.
Ex: Input $I = G$ - graph, $L = \text{graphs with Hamiltonian cycles}$, the witness is the cycle $v_1 - \ldots - v_m - v_1$

$\mathbf{P}$ vs $\mathbf{NP}$ Millennium problem: Is $P \neq NP$?
Computational Complexity

Decision problems: Input $I$ $\rightarrow$ Algorithm: is $I \in L$? $\rightarrow$ Yes/No

Ex: $L = \text{Primes}$, $I$ – an integer
Input size($I$) = $n$ bits

$P = \text{yes/no answer in time } O(n^k)$ some fixed $k$.
Ex: PRIMES, Linear Programming, Graph connectivity ...

$NP = \text{“yes” can be verified in } O(n^k) \text{ for some fixed } k$, i.e. there is a “poly-time witness”.
Ex: Input $I = G$ - graph, $L$=graphs with Hamiltonian cycles, the witness is the cycle $v_1 - \ldots - v_m - v_1$

Counting problems: Input $I$ $\rightarrow$ Algorithm $\rightarrow$ $|C(I)|$
Computational Complexity

**Decision problems:**

\[ \text{Input } I \rightarrow \text{Algorithm: is } I \in L? \rightarrow \text{Yes/No} \]

Ex: \( L = \text{Primes} \), \( I \) – an integer

Input \( \text{size}(I) = n \) bits

\( P \) = yes/no answer in time \( O(n^k) \) some fixed \( k \).

Ex: \( \text{PRIMES, Linear Programming, Graph connectivity} \) ...

\( \text{NP} \) = “yes” can be verified in \( O(n^k) \) for some fixed \( k \), i.e. there is a “poly-time witness”.

Ex: Input \( I = G \) - graph, \( L = \) graphs with Hamiltonian cycles, the witness is the cycle \( v_1 \ldots v_m v_1 \)

**Counting problems:**

\[ \text{Input } I \rightarrow \text{Algorithm} \rightarrow |C(I)| \]

\( \text{FP} \) = answer in time \( O(n^k) \) some fixed \( k \).

Ex: \( \text{Determinant, Spanning trees, recursions} \) ...

\( \text{#P} \) = \#\{\( y : \text{size}(y) < n^k, M(I, y) = 1 \}\} for some fixed \( k \) and \( M \in P \).

\[ = \sum_{y \in \{0,1\}^{n^k}} M(I, y) \]

Ex: Input \( I = G \) - graph, output – number of Hamiltonian cycles in \( G \).
Combinatorial Interpretation via Computational Complexity

Counting and characterizing combinatorial objects given input data $I$

Solve: is $I \in L$, compute $|C(I)|$

"Nice formula"

Product formulas, determinants etc

The problem is in $P$

$\lambda/\mu$, $s_{\lambda/\mu}(1,1,\ldots,1)$

Positive combinatorial interpretation

Ex: Littlewood-Richardson rule

The problem is in $\#P$

$\chi_{\lambda/\mu}(\alpha) = \sum_{T \in \text{SSYT}(\lambda/\mu)} \text{word}(T)$ – ballot sequence?

No "combinatorial interpretation"

Kroneckers, plethysms?

$|\chi_{\lambda}(\alpha)|$

The problem is not in $\#P$

ComputeCharSq $\not\in \#P$...
**Combinatorial Interpretation via Computational Complexity**

| Counting and characterizing combinatorial objects given input data $I$ | Solve: is $I \in L$, compute $|C(I)|$ |
|---|---|
| **“Nice formula”**  
Product formulas, determinants etc | The problem is in P, FP  
$f^{\lambda/\mu}$, $s^{\lambda/\mu}(1,1,\ldots,1)$ |

Kroneckers, plethysms?  
$(\chi^{\lambda}(\alpha))^2$ or $|\chi^{\lambda}(\alpha)|$  
The problem is not in #P  
$\text{ComputeCharSq} \not\in \#P...$
Combinatorial Interpretation via Computational Complexity

Counting and characterizing combinatorial objects given input data $I$

Solve: is $I \in L$, compute $|C(I)|$

“Nice formula”
Product formulas, determinants etc

The problem is in P, FP

$f^{\lambda/\mu}$, $s^{\lambda/\mu}(1, 1, \ldots, 1)$

Positive combinatorial interpretation
Ex: Littlewood-Richardson rule

The problem is in #P

$$c^{\lambda}_{\mu \nu} = \sum_{T \in \text{SSYT}(\lambda/\mu)} \mathbb{1}_{\text{word}(T) - \text{ballot sequence?}}$$
Combinatorial Interpretation via Computational Complexity

Counting and characterizing combinatorial objects given input data \( I \)

Solve: is \( I \in L \), compute \( |C(I)| \)

“Nice formula”
Product formulas, determinants etc

The problem is in \( P, FP \)
\( f^{\lambda/\mu}, s^{\lambda/\mu}(1,1,\ldots,1) \)

Positive combinatorial interpretation
Ex: Littlewood-Richardson rule

The problem is in \( \#P \)
\( c_{\mu\nu}^\lambda = \sum_{T \in SSYT(\lambda/\mu)} 1_{\text{word}(T)\text{—ballot sequence?}} \)

No ”combinatorial interpretation”
Kroneckers, plethysms?
\( (\chi^\lambda(\alpha))^2 \) or \( |\chi^\lambda(\alpha)| \)

The problem is not in \( \#P \)
ComputeCharSq \( \not\in \#P \)...
Characters of $S_n$

characters: $\text{char } S_\lambda = \chi^\lambda : S_n \to \mathbb{C}$

$\chi^\lambda[\alpha]$ – value at permutation of cycle type $\alpha = (\alpha_1, \alpha_2, \ldots)$

Murnagahan–Nakayama rule:

$$\chi^\lambda[\alpha] = \sum_{T : \text{MN tableaux, shape } \lambda, \text{ content } \alpha} (-1)^{ht(T)}$$

— a M-N tableau $T$ of shape $\lambda = (7, 6, 5)$, content $\alpha = (4, 4, 5, 5)$,

$ht(T) = (2 - 1) + (2 - 1) + (3 - 1) + (3 - 1) = 6.$
### Characters of $S_n$

<table>
<thead>
<tr>
<th></th>
<th>$id$</th>
<th>$(1, 2)$</th>
<th>$(1, 2)(3, 4)$</th>
<th>$(1, 2, 3)$</th>
<th>$(1, 2, 3, 4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi^{(4)}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi^{(1,1,1,1)}$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi^{(3,1)}$</td>
<td>3</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi^{(2,1,1)}$</td>
<td>3</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\chi^{(2,2)}$</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>
### Characters of $S_n$

<table>
<thead>
<tr>
<th></th>
<th>id</th>
<th>(1, 2)</th>
<th>(1, 2)(3, 4)</th>
<th>(1, 2, 3)</th>
<th>(1, 2, 3, 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi^{(4)}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi^{(1,1,1,1)}$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi^{(3,1)}$</td>
<td>3</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi^{(2,1,1)}$</td>
<td>3</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\chi^{(2,2)}$</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>

$$\sum_{\lambda \vdash n} \chi^\lambda(id)^2 = n!$$
## Characters of $S_n$

<table>
<thead>
<tr>
<th></th>
<th>id</th>
<th>$(1, 2)$</th>
<th>$(1, 2)(3, 4)$</th>
<th>$(1, 2, 3)$</th>
<th>$(1, 2, 3, 4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi^{(4)}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi^{(1,1,1,1)}$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi^{(3,1)}$</td>
<td>3</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi^{(2,1,1)}$</td>
<td>3</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\chi^{(2,2)}$</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>

$$\sum_{\lambda \vdash n} \chi^\lambda(id)^2 = n!$$

$$\begin{pmatrix} 1 & 2 & 4 \\ 3 & 4 & 2 \end{pmatrix} \xleftarrow{RSK} 4123$$
Characters of $S_n$

<table>
<thead>
<tr>
<th></th>
<th>id</th>
<th>$(1,2)$</th>
<th>$(1,2)(3,4)$</th>
<th>$(1,2,3)$</th>
<th>$(1,2,3,4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi^{(4)}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi^{(1,1,1,1)}$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi^{(3,1)}$</td>
<td>3</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi^{(2,1,1)}$</td>
<td>3</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\chi^{(2,2)}$</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ \sum_{\lambda \vdash n} \chi^\lambda (id)^2 = n! \]

\[
\binom{1 \ 2 \ 4}{3} \quad \binom{1 \ 2 \ 3}{4} \xleftarrow{RSK} 4123
\]

\[ \sum_{\lambda \vdash n} \chi^\lambda (w)^2 = \prod_i i^{c_i} c_i! \]

where $c_i =$ number of cycles of length $i$ in $w \in S_n$. 
Characters of $S_n$

<table>
<thead>
<tr>
<th></th>
<th>id</th>
<th>(1, 2)</th>
<th>(1, 2)(3, 4)</th>
<th>(1, 2, 3)</th>
<th>(1, 2, 3, 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi^{(4)}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi^{(1,1,1,1)}$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi^{(3,1)}$</td>
<td>3</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi^{(2,1,1)}$</td>
<td>3</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\chi^{(2,2)}$</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>

$$\sum_{\lambda \vdash n} \chi^\lambda (id)^2 = n!$$

$$\begin{pmatrix} 1 & 2 & 4 \\ 3 & 4 & 2 \end{pmatrix} \stackrel{RSK}{\mapsto} 4123$$

$$\sum_{\lambda \vdash n} \chi^\lambda (w)^2 = \prod_i i^{c_i} c_i!$$

where $c_i =$ number of cycles of length $i$ in $w \in S_n$.

**ComputeCharSq:**

**Input:** $\lambda, \alpha \vdash n$, unary.

**Output:** the integer $\chi^\lambda (\alpha)^2$. 

Greta Panova
Characters of $S_n$

<table>
<thead>
<tr>
<th>Character</th>
<th>id</th>
<th>(1, 2)</th>
<th>(1, 2)(3, 4)</th>
<th>(1, 2, 3)</th>
<th>(1, 2, 3, 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi^{(4)}$</td>
<td>1 1 1 1</td>
<td>1 1 1 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\chi^{(1,1,1,1)}$</td>
<td>1 -1 1 1</td>
<td>1 -1 1 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\chi^{(3,1)}$</td>
<td>3 1 -1 0</td>
<td>0 -1 1 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\chi^{(2,1,1)}$</td>
<td>3 -1 -1 0</td>
<td>1 -1 1 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\chi^{(2,2)}$</td>
<td>2 0 2 -1</td>
<td>0 0 1 1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$\sum_{\lambda \vdash n} \chi^\lambda(id)^2 = n!$

$\begin{pmatrix} 1 & 2 & 4 \\ 3 & 1 & 2 \\ 4 & 3 & 1 \end{pmatrix} \xrightarrow{RSK} 4123$

$\sum_{\lambda \vdash n} \chi^\lambda(w)^2 = \prod_i i^{c_i} c_i!$

where $c_i = \text{number of cycles of length } i \text{ in } w \in S_n$.

**ComputeCharSq**:
*Input:* $\lambda, \alpha \vdash n$, unary.
*Output:* the integer $\chi^\lambda(\alpha)^2$.

**Theorem (Ikenmeyer-Pak-P’22)**
*ComputeCharSq* $\not\in \#P$ unless $PH = \Sigma^P_2$.

No nice combinatorial interpretation for $\chi^\lambda(\alpha)^2$
Set partitions

Ordered set partitions of items $a = (a_1, \ldots, a_m)$ into bins of sizes $b = (b_1, \ldots, b_k)$:

$$P(a, b) := \#\{(B_1, B_2, \ldots, B_k) : B_1 \sqcup B_2 \sqcup \ldots \sqcup B_k = [m], \sum_{i \in B_j} a_i = b_j \text{ for all } j = 1, \ldots, k\}$$

$$P((2, 4, 5, 7), (9, 9)) = \#\{\{(1, 4\}, \{2, 3\}, \{(2, 3\}, \{1, 4\}\}) = 2$$
Set partitions

Ordered set partitions of items \(a = (a_1, \ldots, a_m)\) into bins of sizes \(b = (b_1, \ldots, b_k)\):

\[
P(a, b) := \# \{(B_1, B_2, \ldots, B_k) : B_1 \sqcup B_2 \sqcup \ldots \sqcup B_k = [m], \sum_{i \in B_j} a_i = b_j \text{ for all } j = 1, \ldots, k\}
\]

\[
P((2, 4, 5, 7), (9, 9)) = \# \{\{(1, 4), \{2, 3\}\}, \{(2, 3), \{1, 4\}\}\} = 2
\]

Jacobi-Trudi/Frobenius character formula:

\[
\chi^\lambda[\alpha] = \sum_{\sigma \in S_k} \text{sgn}(\sigma) P(\alpha, \lambda + \sigma - \text{id})
\]
**Set partitions**

**Ordered set partitions** of items $a = (a_1, \ldots, a_m)$ into bins of sizes $b = (b_1, \ldots, b_k)$:

$$P(a, b) := \# \{(B_1, B_2, \ldots, B_k) : B_1 \sqcup B_2 \sqcup \ldots \sqcup B_k = [m], \sum_{i \in B_j} a_i = b_j \text{ for all } j = 1, \ldots, k\}$$

$$P((2, 4, 5, 7), (9, 9)) = \# \{\{(1, 4), \{2, 3\}\}, \{(2, 3), \{1, 4\}\}\} = 2$$

Jacobi-Trudi/Frobenius character formula:

$$\chi^\lambda[\alpha] = \sum_{\sigma \in S_k} \text{sgn}(\sigma)P(\alpha, \lambda + \sigma - \text{id})$$

**Proposition (IPP)**

*Let $c$ and $d$ be two sequences of nonnegative integers, such that $|c| = |d| + 6$. Then there are partitions $\lambda$ and $\alpha$ of size $O(\ell|c|)$, which are affine functions of $c, d$, such that*

$$\chi^\lambda(\alpha) = P(c, d) - P(c, \overline{d}')$$

*where $\overline{d} := (2, 4, d_1, d_2, \ldots)$ and $\overline{d}' := (1, 5, d_1, d_2, \ldots)$.*
3- and 4d Matchings

Proposition (IPP)

For ∀ two independent 3d matching problem instances $E$ and $E'$, $\exists c$ and $d$, such that

$$\#3DM(E) - \#3DM(E') = \frac{1}{\delta} \left( P(c, d) - P(c, d') \right) = \frac{1}{\delta} \chi^\lambda(\alpha).$$

where $\delta$ is a fixed multiplicity factor, number of orderings.
Proposition (IPP)

For ∀ two independent 3d matching problem instances $E$ and $E'$, ∃ $c$ and $d$, such that

$$\#3DM(E) - \#3DM(E') = \frac{1}{\delta} \left( P(c, d) - P(c, d') \right) = \frac{1}{\delta} \chi^\lambda(\alpha).$$

where $\delta$ is a fixed multiplicity factor, number of orderings.
3- and 4d Matchings

Proposition (IPP)

For ∀ two independent 3d matching problem instances $E$ and $E'$, ∃ $c$ and $d$, such that

$$
\#3DM(E) - \#3DM(E') = \frac{1}{\delta} \left( P(c, d) - P(c, d') \right) = \frac{1}{\delta} \chi^\lambda(\alpha).
$$

where $\delta$ is a fixed multiplicity factor, number of orderings.

Vertices $[4] \times [4]$ and hyperedges $J = (1, 1, 2, 2), (2, 2, 1, 1), (2, 2, 2, 1), (3, 3, 3, 3), (4, 4, 4, 4), (2, 1, 1, 2), (2, 1, 2, 3), (3, 2, 3, 1), (4, 3, 1, 3), (1, 4, 4, 4)$
3- and 4d Matchings

Proposition (IPP)

For ∀ two independent 3d matching problem instances $E$ and $E'$, ∃ $c$ and $d$, such that

$$\#3DM(E) - #3DM(E') = \frac{1}{\delta} \left( P(c, d) - P(c, d') \right) = \frac{1}{\delta} \chi^\lambda(\alpha).$$

where $\delta$ is a fixed multiplicity factor, number of orderings.

Vertices $[4] \times [4]$ and hyperedges $J = (1, 1, 2, 2), (2, 2, 1, 1), (2, 2, 2, 1), (3, 3, 3, 3), (4, 4, 4, 4), (2, 1, 1, 2), (2, 1, 2, 3), (3, 2, 3, 1), (4, 3, 1, 3), (1, 4, 4, 4)$

$\rightarrow$ encoded via vectors $[v_1, \ldots, v_{10}]$

$\rightarrow$ items of size $v_1 + v_2r + \cdots + v_{10}r^9$
3- and 4d Matchings

Proposition (IPP)
For all two independent 3d matching problem instances \(E\) and \(E'\), \(\exists c\ and d,\ such\ that\)

\[
\#3\text{DM}(E) - \#3\text{DM}(E') = \frac{1}{\delta} \left( P(c, \overline{d}) - P(c, \overline{d'}) \right) = \frac{1}{\delta} \chi^\lambda(\alpha).
\]

where \(\delta\) is a fixed multiplicity factor, number of orderings.

Vertices \([4] \times [4]\) and hyperedges \(J = (1, 1, 2, 2), (2, 2, 1, 1), (2, 2, 2, 1), (3, 3, 3, 3), (4, 4, 4, 4), (2, 1, 1, 2), (2, 1, 2, 3), (3, 2, 3, 1), (4, 3, 1, 3), (1, 4, 4, 4)\)
→ encoded via vectors \([v_1, \ldots, v_{10}]\)
→ items of size \(v_1 + v_2 r + \cdots + v_{10} r^9\)

Vertix encodings:
\[
\{[0^{i-1}, 1, 0^4, i, 0^4-j, 3] \mid i \in [4], j \in [4]\}
\{[0^{i-1}, 1, 0^4, i, 0^4-j, 3]^{\text{mult}_J(i,j)} \mid i \in [4], j \in [4]\}\]
3- and 4d Matchings

Proposition (IPP)

For ∀ two independent 3d matching problem instances E and E’, ∃ c and d, such that

\[ \#3DM(E) - \#3DM(E') = \frac{1}{\delta} \left( P(c, d) - P(c, d') \right) = \frac{1}{\delta} \chi^\lambda(\alpha). \]

where δ is a fixed multiplicity factor, number of orderings.

Vertices \([4] \times [4]\) and hyperedges \(J = (1, 1, 2, 2), (2, 2, 1, 1), (2, 2, 2, 1), (3, 3, 3, 3), (4, 4, 4, 4), (2, 1, 1, 2), (2, 1, 2, 3), (3, 2, 3, 1), (4, 3, 1, 3), (1, 4, 4, 4)\)

→ encoded via vectors \([v_1, \ldots, v_{10}]\)

→ items of size \(v_1 + v_2r + \cdots + v_{10}r^9\)

Vertex encodings:
\[
\{[0^{i-1}, 1, 0^4, i, 0^4-j, 3] \mid i \in [4], j \in [4]\} \\
\{[0^{i-1}, 1, 0^4, i, 0^4-j, 3]^{\text{mult}_J(i,j)} \mid i \in [4], j \in [4]\}
\]

Hyperedge \((1, 1, 2, 2)\)

→ \([0^4, 1, 4-1, 4-1, 4-2, 4-2, 0]\)

Bins size \(b_1 = [1^5, 4^4, 12]\), bins: \(b = (b_1^{10})\):
3- and 4d Matchings

Proposition (IPP)

For ∀ two independent 3d matching problem instances \( E \) and \( E' \), \( \exists c \) and \( d \), such that

\[
\#3DM(E) - \#3DM(E') = \frac{1}{\delta} \left( P(c, d) - P(c, d') \right) = \frac{1}{\delta} \chi^\lambda(\alpha).
\]

where \( \delta \) is a fixed multiplicity factor, number of orderings.

Vertices \([4] \times [4]\) and hyperedges \( J = (1, 1, 2, 2), (2, 2, 1, 1), (2, 2, 2, 1), (3, 3, 3, 3), (4, 4, 4, 4), (2, 1, 1, 2), (2, 1, 2, 3), (3, 2, 3, 1), (4, 3, 1, 3), (1, 4, 4, 4)\)

→ encoded via vectors \([v_1, \ldots, v_{10}]\)

→ items of size \(v_1 + v_2 r + \cdots + v_{10} r^9\)

Vertix encodings:
\[
\{[0^{i-1} 1, 0^4, i, 0^{4-j} 3] \mid i \in [4], j \in [4]\}
\]
\[
\{[0^{i-1} 1, 0^4, i, 0^{4-j} 3]^{\text{mult}_{J(i,j)}} \mid i \in [4], j \in [4]\}
\]

Hyperedge \((1, 1, 2, 2)\)

→ \([0^4, 1, 4 - 1, 4 - 1, 4 - 2, 4 - 2, 0]\)

Bins size \(b_1 = [1^5, 4^4, 12]\), bins: \(b = (b_1^{10}):\)

\[
[0, 0, 0, 0, 1, 3, 3, 2, 2, 0] + [1, 0, 0, 0, 0, 1, 0, 0, 0, 3] + [0, 1, 0, 0, 0, 0, 1, 0, 0, 3] + [0, 0, 1, 0, 0, 0, 2, 0, 3] + [0, 0, 0, 1, 0, 0, 0, 2, 3] = [1, 1, 1, 1, 4, 4, 4, 4, 12]
\]
Characters are as hard as the polynomial hierarchy

Theorem (Ikenmeyer-Pak-P’22)
Let \( \chi^2 : (\lambda, \pi) \mapsto (\chi^\lambda(\pi))^2 \), where \( \lambda \vdash n \) and \( \pi \in S_n \). If \( \chi^2 \in \#P \), then the polynomial hierarchy collapses to the second level: \( \text{PH} = \Sigma^P_2 = \text{NP} \).

Polynomial hierarchy: \( \Sigma^0_0 = \text{P}, \Sigma^i_{i+1} = \text{NP} \Sigma^i_p, \text{PH} = \bigcup_{i=0}^\infty \Sigma^i_p. \)

\[1\] A hypothesis widely believed to be false, similar to \( P \neq \text{NP} \).
Characters are as hard as the polynomial hierarchy

**Theorem (Ikenmeyer-Pak-P’22)**

Let \( \chi^2 : (\lambda, \pi) \mapsto (\chi^\lambda(\pi))^2 \), where \( \lambda \vdash n \) and \( \pi \in S_n \). If \( \chi^2 \in \#P \), then the polynomial hierarchy collapses to the second level: \( PH = \Sigma^p_2 = NP \). \(^1\)

Polynomial hierarchy: \( \Sigma^p_0 = P, \Sigma^p_{i+1} = NP \Sigma^p_i \), \( PH = \bigcup_{i=0}^\infty \Sigma^p_i \).

\[
#3DM(E) - #3DM(E') = \frac{1}{\delta} \chi^\lambda(\alpha)
\]

\[
\implies [\chi = 0] \text{ is } C_{=}P := [\text{GapP } = 0]-\text{complete.}
\]

\(^1\)A hypothesis widely believed to be false, similar to \( P \neq NP \).
Characters are as hard as the polynomial hierarchy

Theorem (Ikenmeyer-Pak-P’22)
Let $\chi^2 : (\lambda, \pi) \mapsto (\chi^\lambda(\pi))^2$, where $\lambda \vdash n$ and $\pi \in S_n$. If $\chi^2 \in \#P$, then the polynomial hierarchy collapses to the second level: $\text{PH} = \Sigma_2^p = \text{NP}$.

Polynomial hierarchy: $\Sigma_0^p = P$, $\Sigma_{i+1}^p = \text{NP} \Sigma_i^p$, $\text{PH} = \bigcup_{i=0}^{\infty} \Sigma_i^p$.

$\#3DM(E) - \#3DM(E') = \frac{1}{d} \chi^\lambda(\alpha)$

$\implies [\chi = 0]$ is $C = \text{P} := [\text{GapP} = 0]$-complete.

If $\chi^2 \in \#P \implies [\chi^2 > 0] \in \text{NP}$, so $[\chi \neq 0] \in \text{NP}$ and hence $[\chi = 0] \in \text{coNP}$.

$^1$A hypothesis widely believed to be false, similar to $P \neq \text{NP}$
Characters are as hard as the polynomial hierarchy

**Theorem (Ikenmeyer-Pak-P’22)**

Let $\chi^2 : (\lambda, \pi) \mapsto (\chi^\lambda(\pi))^2$, where $\lambda \vdash n$ and $\pi \in S_n$. If $\chi^2 \in \#P$, then the polynomial hierarchy collapses to the second level: $\text{PH} = \Sigma^p_2 = \text{NP}$.

Polynomial hierarchy: $\Sigma^p_0 = \text{P}$, $\Sigma^p_{i+1} = \text{NP} \Sigma^p_i$, $\text{PH} = \bigcup_{i=0}^\infty \Sigma^p_i$.

$$\#3\text{DM}(E) - \#3\text{DM}(E') = \frac{1}{\delta} \chi^\lambda(\alpha)$$

$\implies [\chi = 0]$ is $C = \text{P} := [\text{GapP} = 0]$-complete.

If $\chi^2 \in \#\text{P} \implies [\chi^2 > 0] \in \text{NP}$, so $[\chi \neq 0] \in \text{NP}$ and hence $[\chi = 0] \in \text{coNP}$.

$\implies C = \text{P} \subset \text{coNP}$

$\implies$ since $\text{PH} \subset \text{NP}^{C = \text{P}}$ (Tarui’91) then $\text{PH} \subset \text{NP}^{\text{coNP}}$, so $\text{PH} = \Sigma^p_2$.

---

1A hypothesis widely believed to be false, similar to $\text{P} \neq \text{NP}$
The End

Computing Kronecker, plethysm coefficients and especially $S_n$ characters...

Thank you for your attention!