Symmetric group characters are computationally hard

$\chi^\lambda(\alpha)^2$ or $|\chi^\lambda(\alpha)|$ do NOT have a nice combinatorial interpretation!

Greta Panova
joint work with Christian Ikenmeyer and Igor Pak

University of Southern California

FPSAC 2023

Symmetric group representations

S_n – permutations under composition:

$$\pi : [1, 2, \ldots, n] \xrightarrow{\sim} [1, 2, \ldots, n], \qquad \pi\sigma = \pi(\sigma)$$

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Example: if $V = \mathbb{C}^3$, $\pi \in S_3$, set $\pi(e_i) := e_{\pi_i}$ for $i = 1..3$, so e.g. $231 \to \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

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The **irreducible representations** of the **symmetric group** S_n: the *Specht modules* $\mathbb{S}_\lambda$

$$V = \underbrace{\mathbb{C}\langle e_1 + e_2 + e_3 \rangle}_{V_1} \oplus \underbrace{\mathbb{C}\langle e_1 - e_2, e_2 - e_3 \rangle}_{V_2}$$

$$\mathbb{S}_{(3)} \simeq V_1 \qquad \mathbb{S}_{(2,1)} \simeq V_2$$

Basis indexed by SYTs of shape λ, so $\dim \mathbb{S}_\lambda = f^\lambda := \#\{T : \text{ SYT, shape } \lambda\}$.

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Characters: $\quad \operatorname{char} \mathbb{S}_\lambda = \chi^\lambda : S_n \to \mathbb{C}$

$$\chi^{(2,1)}(231) = \mathit{Trace} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = -1$$

Structure constants

Tensor product of irreducible GL representations (Weyl modules V_α):

$$V_\lambda \otimes V_\mu = \oplus_\nu V_\nu^{\oplus c^\nu_{\lambda\mu}}$$

Littlewood-Richardson coefficients: $c^\nu_{\lambda\mu} = \langle \chi^\lambda \times \chi^\mu, \chi^\nu \downarrow^{S_n}_{S_k \times S_{n-k}} \rangle$

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The coefficient $c^\nu_{\lambda\mu}$ is equal to the number of LR tableaux of shape ν/μ and type λ.

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Major problems in Algebraic Combinatorics

[Murnaghan, 1938]: $c^{\lambda}_{\mu\nu} = g\left((N-|\lambda|,\lambda),(N-|\mu|,\mu),(N-|\nu|,\nu)\right)$ for $|\lambda| = |\mu| + |\nu|$ and N-large.

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Find a positive combinatorial interpretation for $g(\lambda, \mu, \nu)$, i.e. a family of combinatorial objects $\mathcal{O}_{\lambda,\mu,\nu}$, s.t. $g(\lambda, \mu, \nu) = \#\mathcal{O}_{\lambda,\mu,\nu}$.

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What is really a "combinatorial interpretation"? → NEXT

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Decision problems: Input I $\longrightarrow$ Algorithm: is $I \in L$? $\longrightarrow$ Yes/No

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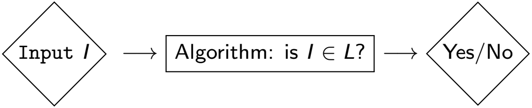

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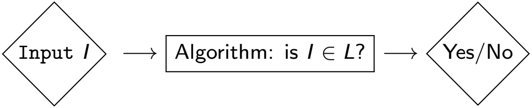

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P vs NP Millennium problem: Is P $\neq$ NP?

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FP = answer in time $O(n^k)$ some fixed k.
Ex: Determinant, Spanning trees, recursions ...

#P $= \#\{y : size(y) < n^k, M(I, y) = 1\}$ for some fixed k and $M \in$ P.
$= \sum_{y \in \{0,1\}^{n^k}} M(I, y)$
Ex: Input $I = G$ - graph, output – number of Hamiltonian cycles in G.

Combinatorial Interpretation via Computational Complexity

Counting and characterizing combinatorial objects given input data I	Solve: is $I \in L$, compute $|C(I)|$

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No "combinatorial interpretation" Kroneckers, plethysms? $(\chi^\lambda(\alpha))^2$ or $\|\chi^\lambda(\alpha)\|$	The problem is not in #P ComputeCharSq$\notin$ #P...

Characters of S_n

characters: $\quad \text{char}\, \mathbb{S}_\lambda = \chi^\lambda : S_n \to \mathbb{C}$

$\chi^\lambda[\alpha]$ – value at permutation of cycle type $\alpha = (\alpha_1, \alpha_2, \ldots)$

Murnaghan–Nakayama rule:

$$\chi^\lambda[\alpha] = \sum_{T \,:\, \text{MN tableaux, shape } \lambda, \text{ content } \alpha} (-1)^{ht(T)}$$

1	1	1	3	3	4	4
1	2	2	3	4	4	
2	2	3	3	4		

— a M-N tableau T of shape $\lambda = (7, 6, 5)$,
content $\alpha = (4, 4, 5, 5)$,
$ht(T) = (2-1) + (2-1) + (3-1) + (3-1) = 6.$

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$$\sum_{\lambda \vdash n} \chi^{\lambda}(id)^2 = n!$$

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Input: $\lambda, \alpha \vdash n$, unary.
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Theorem (Ikenmeyer-Pak-P'22)

ComputeCharSq $\notin$ #P unless $PH = \Sigma_2^{\mathsf{P}}$.

No nice combinatorial interpretation for $\chi^\lambda(\alpha)^2$

Set partitions

Ordered set partitions of items $\mathbf{a} = (a_1, \ldots, a_m)$ into bins of sizes $\mathbf{b} = (b_1, \ldots, b_k)$:

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Proposition (IPP)

Let $\mathbf{c}$ and $\mathbf{d}$ be two sequences of nonnegative integers, such that $|\mathbf{c}| = |\mathbf{d}| + 6$. Then there ara partitions λ and α of size $O(\ell|\mathbf{c}|)$, which are affine functions of $\mathbf{c}, \mathbf{d}$, such that

$$\chi^\lambda(\alpha) = P(\mathbf{c}, \overline{\mathbf{d}}) - P(\mathbf{c}, \overline{\mathbf{d}'}),$$

where $\overline{\mathbf{d}} := (2, 4, d_1, d_2, \ldots)$ and $\overline{\mathbf{d}'} := (1, 5, d_1, d_2, \ldots)$.

3- and 4d Matchings

Proposition (IPP)

For $\forall$ two independent 3d matching problem instances E and E', $\exists$ $\mathbf{c}$ and $\mathbf{d}$, such that

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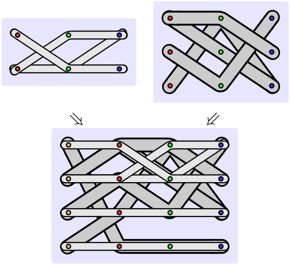

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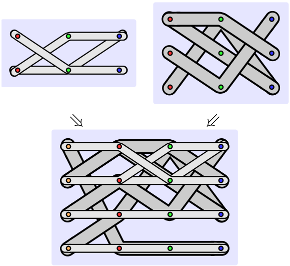

Vertices $[4] \times [4]$ and hyperedges $J =$ $(1,1,2,2)$,$(2,2,1,1)$,$(2,2,2,1)$,$(3,3,3,3)$,$(4,4,4,4)$, $(2,1,1,2)$,$(2,1,2,3)$,$(3,2,3,1)$,$(4,3,1,3)$,$(1,4,4,4)$

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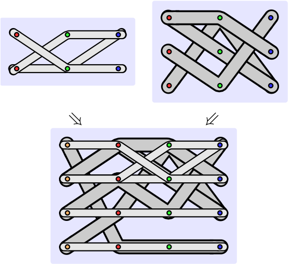

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→ encoded via vectors $[v_1, \ldots, v_{10}]$
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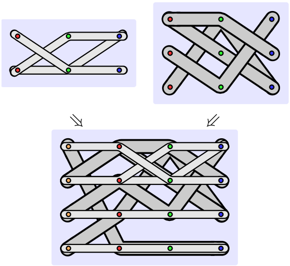

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$\{[0^{j-1},1,0^4,i,0^{4-j},3] \mid i\in[4], j\in[4]\}$
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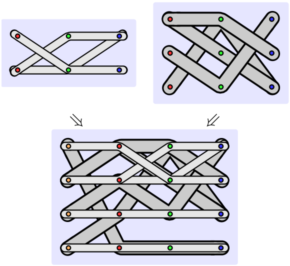

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Hyperedge $(1,1,2,2)$
$\rightarrow [0^4, 1, 4-1, 4-1, 4-2, 4-2, 0]$
Bins size $b_1 = [1^5, 4^4, 12]$, bins: $\mathbf{b} = (b_1^{10})$:

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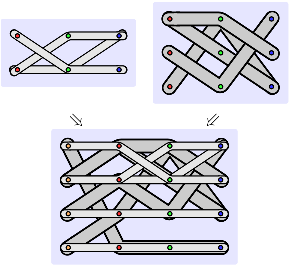

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$$+[0,0,1,0,0,0,0,2,0,3] + [0,0,0,1,0,0,0,0,2,3] = [1,1,1,1,1,4,4,4,4,12]$$

Characters are as hard as the polynomial hierarchy

Theorem (Ikenmeyer-Pak-P'22)

Let $\chi^2 : (\lambda, \pi) \mapsto \left(\chi^\lambda(\pi)\right)^2$, *where* $\lambda \vdash n$ *and* $\pi \in S_n$. *If* $\chi^2 \in \#\mathsf{P}$, *then the polynomial hierarchy collapses to the second level:* $\mathsf{PH} = \Sigma_2^{\mathsf{p}} = \mathsf{NP}$ [1].

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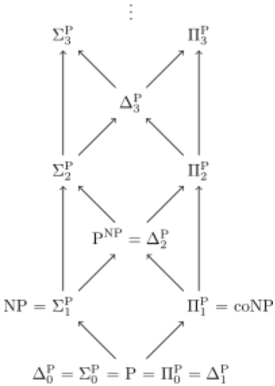

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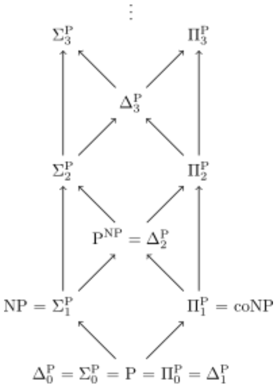

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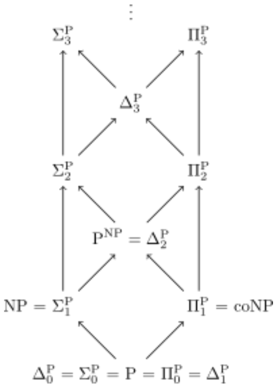

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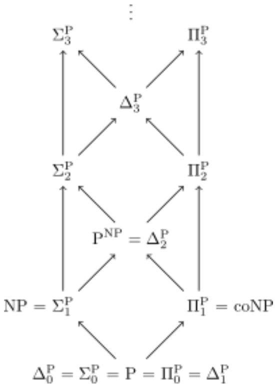

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$\Longrightarrow \mathrm{C_=P} \subset \mathrm{coNP}$

$\Longrightarrow$ since $\mathrm{PH} \subset \mathrm{NP}^{\mathrm{C_=P}}$ (Tarui'91) then $\mathrm{PH} \subset \mathrm{NP}^{\mathrm{coNP}}$, so $\mathrm{PH} = \Sigma_2^{\mathrm{p}}$ □

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The End

Computing Kronecker, plethysm coefficients and especially S_n characters...

Thank you for your attention!