Recursions and Proofs in Cataland

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joint with

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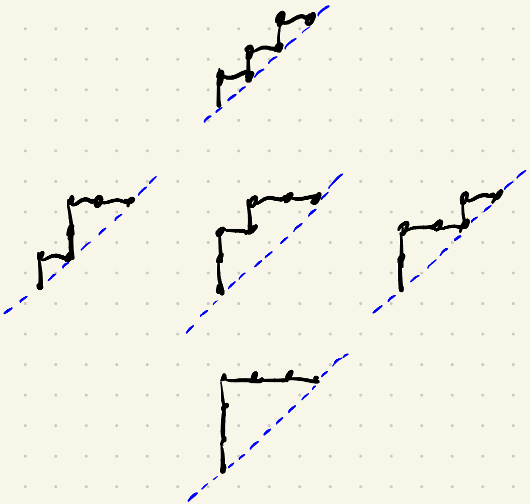
Delivered @ the hottest FPSAC on record

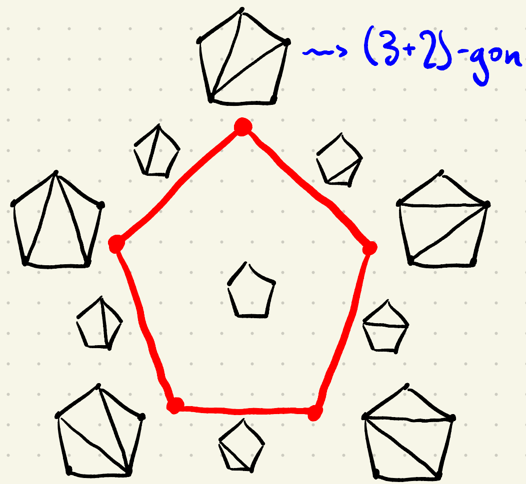
@ UCDavis Jul. 18th 2023

Attractions at the Catalan Zoo

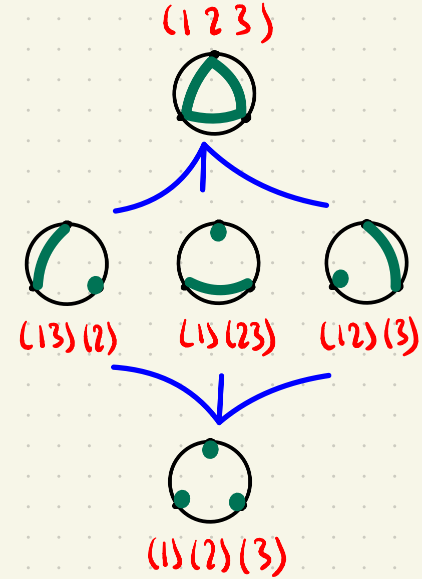
The Catalan numbers $Cat(n) := \frac{1}{n+1}\binom{2n}{n}$ count:



Dyck Paths $(0,0) \to (n,n)$



Vertices of the associahedron



Noncrossing Partitions

Cat(n): 1, 1, 2, 5, 14, 42, 132, 429, ...

AKA: the answer to the ultimate question of Life the Universe, and Everything

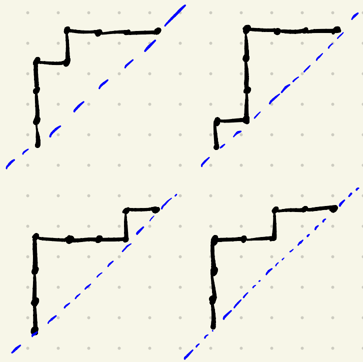
Germain Kreweras visits the Catalan Zoo

The Kreweras numbers $\mathrm{Krew}(\lambda) := \dfrac{n\cdot(n-1)\cdots(n-k+2)}{\mathrm{Sym}(\lambda)}$ count:

$\lambda \vdash n$

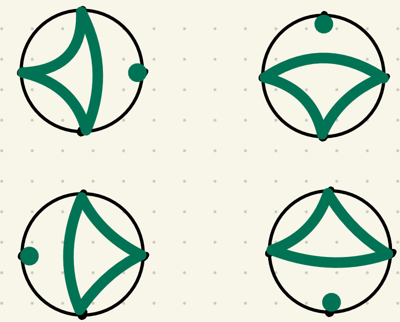
$\lambda = (\lambda_1, \dots, \lambda_k) = (1^{m_1}, \dots, n^{m_n})$

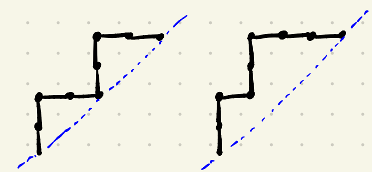
$\mathrm{Sym}(\lambda) := m_1! \cdots m_n!$



$\lambda = (3,1)$

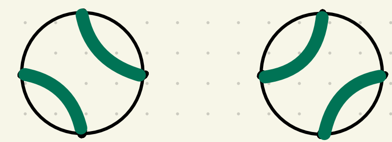
$\mathrm{Krew}(\lambda) = 4 = \dfrac{4}{1}$





$\lambda = (2,2)$

$\mathrm{Krew}(\lambda) = 2 = \dfrac{4}{2}$

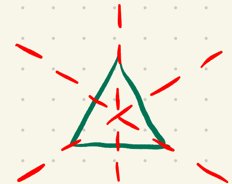


Dyck Paths whose vertical runs determine a partition λ

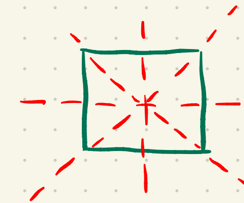
Noncrossing Partitions whose blocks determine a partition λ

Krew(λ)	1	4	2	6	1	$\Rightarrow$ 14
λ	(4)	$(3,1)$	$(2,2)$	$(2,1,1)$	$(1,1,1,1)$	

From S_n to Reflection Groups W



$I_2(3) \simeq A_2$



$I_2(4) \simeq B_2$

Reflection groups are...
Finite subgroups $W \leq GL(\mathbb{R}^n)$
generated by Euclidean reflections!

Reflection groups are...
classified in four infinite families $\underset{\text{}}{\overset{S_{n+1}}{A_n}}$, $\underbrace{B_n}_{\text{hyperoctahedral}}$, D_n, and $\underbrace{I_2(m)}_{\text{dihedrals}}$

and some exceptionals ($H_3, H_4, F_4, E_6, E_7, E_8$)

Proofs are often case-by-case!

Reflection groups have...

- nice presentations $W = \langle \underbrace{s_1, \ldots, s_n}_{\text{simple gen's}} \mid s_i^2 = (s_i s_j)^{m_{ij}} = 1 \rangle$

- Coxeter elements $c = s_1 \cdot s_2 \cdots s_n$ order$(c) = h \rightarrow$ "Coxeter number"

From S_n to Reflection Groups W: Partitions

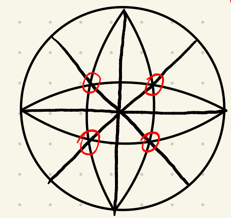
A partition

$$\{\{1,3,4\},\{2,6,7,8\},\{5\}\}$$

determines an intersection of hyperplanes: a flat $X \in \mathcal{L}_W$

$$x_1 = x_3 \cap x_3 = x_4 \cap x_2 = x_6 \cap x_6 = x_7 \cap x_7 = x_8$$

B_3



$\otimes : A_2 \simeq S_3$

W-orbits of flats $[X] \in \mathcal{L}_W / W$ play the role of cycle types

exceptional numerology:

χ ⇝ characteristic polynomial

$\mathcal{A}_W$ ⇝ reflection arrangement

$\mathcal{A}_W^X$ ⇝ restricted arrangement

"group exponents"

$$\chi(\mathcal{A}_W, t) = \prod_{i=1}^{\mathrm{rank}(W)} (t - e_i)$$

in S_n: $e_i = i$

"Orlik-Solomon exponents"

$$\chi(\mathcal{A}_W^X, t) = \prod_{i=1}^{\dim(X)} (t - b_i^X)$$

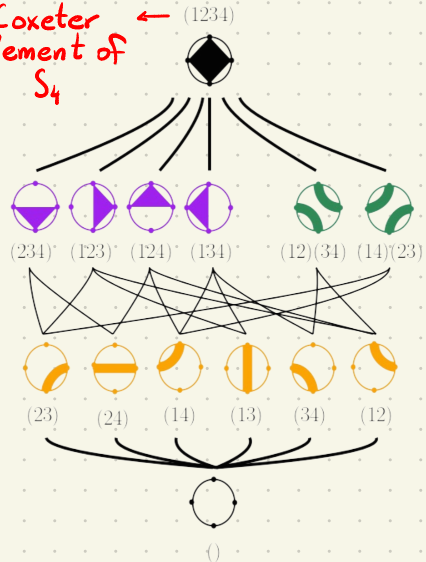
in S_n: $b_i^X = i$

$\Downarrow$

detect copies of V in H^*(Springer fibers)

From S_n to reflection groups W: Noncrossing Partitions

Coxeter element of S_4 ←



- c Coxeter element
- $\leq_R$ order induced by reflection length
- $NC(W) := [1, c]_{\leq_R}$ the W-noncrossing partitions

They are counted by the W-Kreweras numbers

$$m\text{-}\mathrm{Krew}^{NC}_{W}([X]) = \frac{\prod (mh+1-b_i^X)}{[N(X):W_X]}$$

$m=1$ case # of elements in $NC(W)$ of "type" $[X]$

arbitrary m case # of m-chains in $NC(W)$ starting at an element of "type" $[X]$

Compare with

$$\mathrm{Krew}(\lambda) = \frac{n(n-1)\cdots(n-k+2)}{\mathrm{Sym}(\lambda)}$$

$h = n$

$b_i^X = i$

Athanasiadis-Reiner '02
Rhoades '17 } case-by-case ???

Our main contribution: A case-free proof

Theorem [$\underbrace{\text{Ath-Rein '02, Rhoades '17}}_{\text{case-by-case}}$, $\underbrace{\text{D.-Josuat-Vergès '22}}_{\text{case-free}}$]

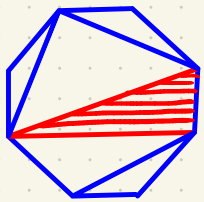
For an irred. finite Coxeter group W, any $m \in \mathbb{Z}_{\geq 0}$ and parabolic type $[X] \in \mathcal{L}_W/W$

we have
$$\underbrace{m\text{-}\mathrm{Krew}_W^{NC}([X])}_{\substack{m\text{-chains in } NC(W) \\ \text{that start at type } [X]}} = \underbrace{\frac{\prod (mh+1-b_i^X)}{[N(X):W_X]}}$$

h: Coxeter number
b_i^X: Orlik-Solomon exponents

Could a recursion work ?!?

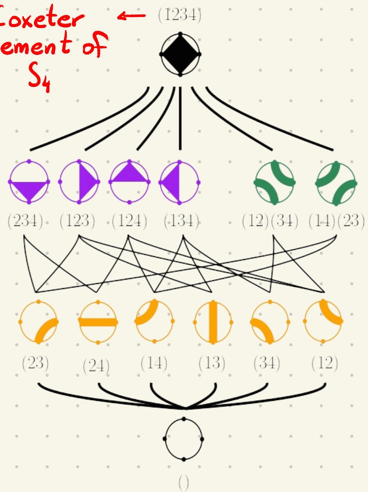
Already $C_n = C_{n-1} \cdot C_0 + \cdots + C_0 \cdot C_{n-1}$
is non-trivial to give $C_n = \frac{1}{n+1}\binom{2n}{n}$



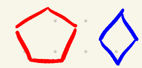
Proof via an (over)abundance of recursions

Recursion on chains

Coxeter element of S_4



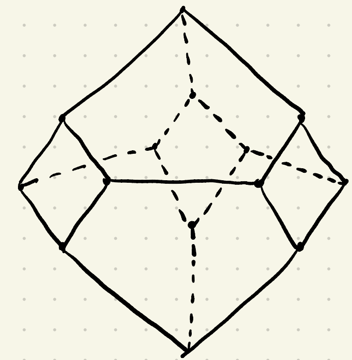
Whitney #s		Face #s
1	← h-to-f transformation →	14
$6 = 4 + 2$		21
6	$\sum h_k (1+t)^k = \sum f_k t^k$	$6 + 3 = 9$
1		1



$$6 + \binom{3}{1} \cdot 1 = 9$$

$4 + 2 \cdot 1 = 6$ $\qquad$ $2 + 1 \cdot 1 = 3$

Recursion on faces



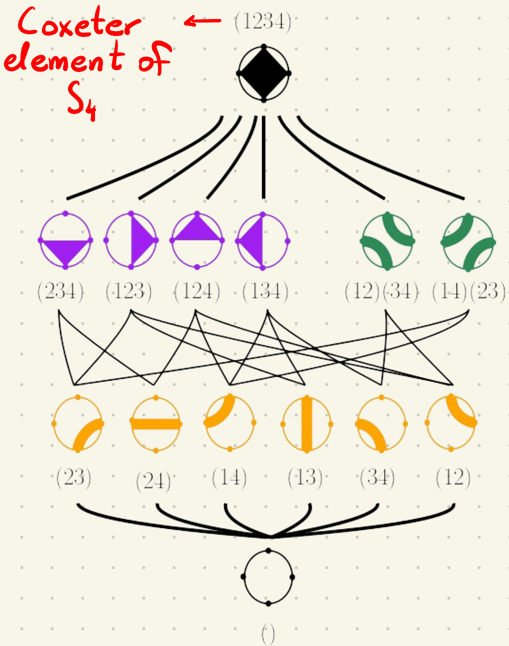
W-associahedron / W-cluster complex

$$m\text{-}\mathrm{Krew}^{NC}_{W}([X]) = \frac{\prod (mh+1-b_i^X)}{[N(X):W_X]}$$

$$m\text{-}\mathrm{Lod}^{NC}_{W}([X]) = \frac{\prod (mh+1+b_i^X)}{[N(X):W_X]}$$

Proof: We show that the W-Kreweras and W-Loday formulas satisfy combinatorial recursions

Structure of the proof: Recursion on chains



Combinatorial recursion on zeta polynomials $\Big\}$ Count chains of length m with respect to their k-th element

$(m = k + r)$

$$m\text{-}\mathrm{Krew}_W^{NC}([x]) = \sum_{[y]} k\text{-}\mathrm{Krew}_{W_y}^{NC}([x]) \cdot r\text{-}\mathrm{Krew}_W^{NC}([y])$$

$["x", c]_{\leq R}$ $\quad$ $["x", "y"]_{\leq R}$ $\quad$ $["y", c]_{\leq R}$

Expanding the formulas

$$m\text{-}\mathrm{Krew}_W^{NC}([x]) = \frac{\prod (mh+1-b_i^x)}{[N(x):W_x]}$$ it becomes

$$\boxed{(t+kh)^{\dim(X)} = \sum_{Y \subset X} t^{\dim(Y)} \cdot \prod_{i=1}^{d(X)-d(Y)} k\, h_i(X,Y)}$$

[D. '21-23 via W-Laplacians]

Parking @ the Catalan Zoo

Haiman '92 considers the diagonal action $S_n \curvearrowright \mathbb{C}[x_1,\ldots,x_n,y_1,y_2,\ldots,y_n] =: \mathbb{C}[\vec{x},\vec{y}]$

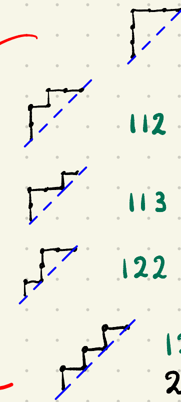
He defines diagonal coinvariants/harmonics $DH_n := \mathbb{C}[\vec{x},\vec{y}] \big/ \langle \mathbb{C}[\vec{x},\vec{y}]_+^{S_n} \rangle$

CONJECTURED '92

HAIMAN

PROVED '02

- $\dim(DH_n) = (n+1)^{n-1}$
- $DH_n \otimes \det \cong_{S_n} \mathbb{Z}_n^{n+1} / \mathbb{Z}_n$
- $\mathrm{Frob}(DH_n) = \nabla(e_n)$



111

112 121 211

113 131 311

122 212 221

123 132 213
231 312 321

Reflection groups W

$DH(W) := \mathbb{C}[\vec{x},\vec{y}] \big/ \langle \mathbb{C}[\vec{x},\vec{y}]_+^{W} \rangle$ often has $\dim(DH(W)) > (h+1)^n = \dim\left(Q/(h+1)Q\right)$

Gordon ['03] constructs a quotient $\mathrm{Park}(W) := DH(W) \big/ \substack{\text{magic} \\ \& \; t = 1/q}$ with correct properties

Parabolic Decompositions of the Parking Space

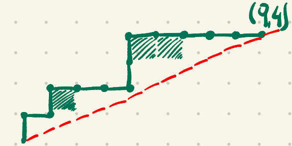
MOSTLY SUBSUMED BY SHUFFLE THEOREM

- $$\nabla^m_{\substack{t=1\\q=1}}(e_n) = \sum_{\lambda \vdash n} \underbrace{\frac{mn(mn-1)\cdots(mn-k+2)}{\mathrm{Sym}(\lambda)}}_{\text{Fuss-Kreweras numbers}} \cdot e_\lambda \qquad (\lambda = \lambda_1, \lambda_2, \dots, \lambda_k)$$

- $$\nabla^m_{q=1}(e_n) = \sum_{\substack{\pi \text{ Dyck Path}\\ (0,0)\to(mn+1,n)}} t^{\mathrm{area}(\pi)} \cdot e_{\lambda(\pi)}$$

  "compatible recursions"

  $\lambda(\pi)$: rise-partition of π



$\mathrm{area}(\pi) = 3$

$\lambda(\pi) = (2,1,1)$

- $$\nabla^m_{t=1/q}(e_n) = q^{-\binom{n}{2}} \cdot \sum_{\lambda \vdash n} \underbrace{\frac{[mn]_q \cdots\cdots [mn-k+2]_q}{[\mathrm{Sym}(\lambda)]_q}}_{\text{encodes symmetries via C.S.P}} \cdot \omega\left(\tilde{H}_\lambda(q)\right)$$

  $\tilde{H}_\lambda$ → Modified Hall-Littlewood

  ↳ $H^*(B_\lambda; q)$ → Springer fiber

  "compute separately"

[Reiner-Sommers]

Parabolic Decompositions of W-Parking spaces $\mathrm{Park}^{\langle m\rangle}(W;q)$

ANCIENT

- $\mathrm{Park}^{\langle m\rangle}(W;q=1) \simeq_W \bigoplus_{[X]} \underbrace{\dfrac{\prod_{i=1}^{\dim(X)}(mh+1-b_i^X)}{[N(X):W_X]}}_{\text{W-Fuss-Kreweras numbers}} \cdot \Big\uparrow_{W_X}^{W} \mathrm{triv}$

  [Orlik-Solomon-Terao, Sommers]

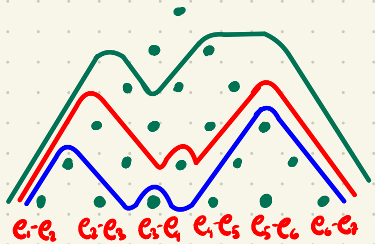
- $\mathrm{Park}^{\langle m\rangle}(W;q=1) \simeq_W \bigoplus_{[X]} \# \left\{ \begin{array}{l} \text{type-}[X] \ m\text{-chains} \\ \text{of order ideals in} \\ \text{the root poset} \end{array} \right\} \cdot \Big\uparrow_{W_X}^{W} \mathrm{triv}$

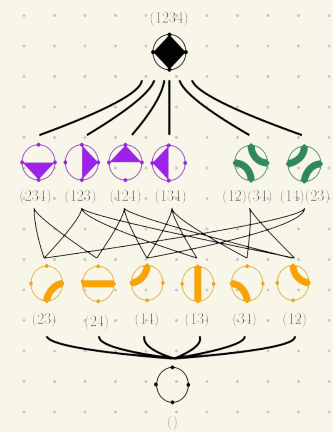
'00 [Cellini-Papi, Sommers, Athanasiadis] CASE-FREE?

- $\mathrm{Park}^{\langle m\rangle}(W;q=1) \simeq_W \bigoplus_{[X]} \# \left\{ \begin{array}{l} \text{type-}[X] \ m\text{-chains} \\ \text{of non-crossing} \\ \text{partitions} \end{array} \right\} \cdot \Big\uparrow_{W_X}^{W} \mathrm{triv}$

'02 [Athanasiadis-Reiner, Rhoades] CASE-BY-CASE

'22 [D.-Josuat-Vergès] CASE-FREE?





Give me the symmetric group version that I know and love

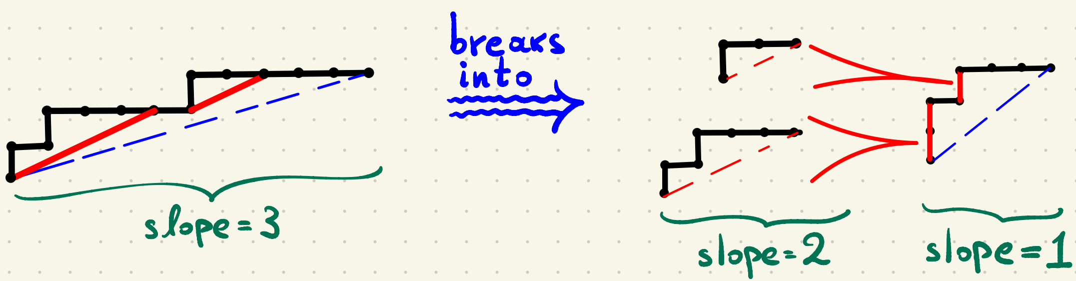
The $m=k+r$ chain decomposition in S_n becomes

$$\boxed{\nabla^{m}_{q=1}(e_n) = \sum_{\substack{\pi \text{ Dyck path} \\ (0,0)\to(kn+1,n)}} t^{\mathrm{area}(\pi)} \cdot \prod_{i=1}^{s} \nabla^{r}_{q=1}(e_{\lambda_i(\pi)})}$$

$\left[\lambda(\pi) = (\lambda_1(\pi), \dots, \lambda_s(\pi))\right.$ "rise partition" of $\left.\pi\right]$

Proof: $\nabla_{q=1}$ is multiplicative

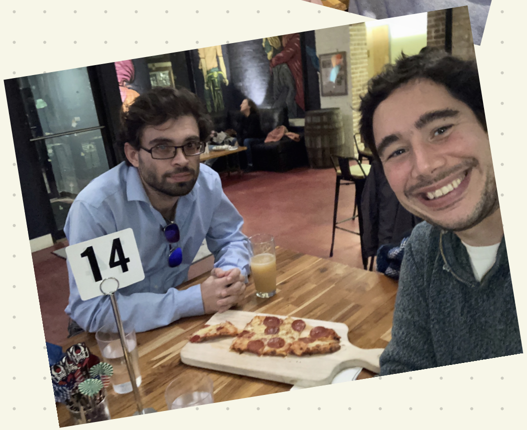
Combinatorially:



Open Problems

- Analogous graded decompositions for other types
- Interaction with braid varieties [GLTW]
- Combinatorial models for some q,t-grading?

Thank You



- Counting nearest faraway flats for Coxeter chambers : 2209.06201 (JCA '23)
- The generalized cluster complex: refined enumeration of spaces 2209.12540 (w/ M. J-V)
- Reflection Laplacians, parking spaces, and multiderivations in Coxeter-Catalan soon?
- Recursions and Proofs in Cataland (w/ M. J-V) soon? preview @ extended abstract

The W-Laplacian and its spectrum

- $W \leq GL(V)$ a finite Coxeter group
- Φ its root system
- Define the W-Laplacian

$$\boxed{L_W(v) := \sum_{\sigma \in \Phi^+} \langle v, \hat{\sigma} \rangle \cdot \sigma \ \in \mathrm{End}(V)}$$

akin to the $L_G := B \cdot B^T$ defn of the graph Laplacian

Its characteristic polynomial:

$$\det(L_W + t) = \sum_{\{\sigma_1, \dots, \sigma_k\} \subseteq \Phi^+} t^{n-k} \cdot \det\left(\langle \sigma_{i_r}, \sigma_{i_s} \rangle\right)_{r,s=1}^{k}$$

$\Downarrow$ group wrt $\mathrm{span}(\sigma_i)$

$$\det(L_W + t) = \sum_{X \in \mathcal{L}_W} t^{\dim(X)} \cdot \mathrm{pdet}(L_{W_X})$$

$\Downarrow$ $\det(L_W) = h^n$

$$(h+t)^n = \sum_{X \in \mathcal{L}_W} t^{\dim(X)} \cdot \prod_{i=1}^{\mathrm{codim}(X)} h_i(W_X)$$

$\Longrightarrow$ Compare with arithmetic characteristic polynomial.

$\Longrightarrow$ Essentially the popular formula

$$h \cdot n = 2 \cdot \#\{\text{reflections of } W\}$$