Tautological classes of matroids

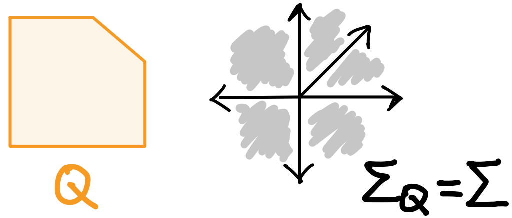
Chris Eur (with Andrew Berget, Hunter Spink, Dennis Tseng)

(1) A tale of two rings (K & H^*)

(2) Matroids

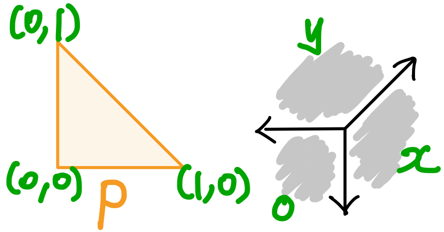
(3) One ring to rule them all...

Σ a projective unimodular fan in $\mathbb{R}^m$.



$P \subset \mathbb{R}^m$ a deformation of Σ

$\hookrightarrow$ a (lattice) polytope whose Σ_P coarsens Σ.



$\rightsquigarrow$ two functions $[P]$ and $h_P : \mathbb{R}^m \longrightarrow \mathbb{R}$

① Indicator fct

$$[P](x) = \begin{cases} 1 & \text{if } x \in P \\ 0 & \text{else} \end{cases}$$

② Support fct (piecewise linear on Σ)

$$h_P(x) = \max_{y \in P} \langle x, y \rangle$$

$$\frac{\mathbb{Z}\{[P] : P \text{ a def. of } \Sigma\}}{\langle [P]-[P+v] : v \in \mathbb{Z}^m \rangle} = K(\Sigma) \qquad\qquad H^*(\Sigma) = \frac{\{\text{piecewise polynom. on } \Sigma\}}{\langle \text{(globally) linear fcts} \rangle}$$

$$[P][P'] = [P+P']$$

$$\begin{array}{ccc} K(\Sigma) & \qquad & H^*(\Sigma) \\ \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \deg_\Sigma} \\ \mathbb{Z} & & \mathbb{Z} \end{array}$$

$$\chi([P]) = \#|P \cap \mathbb{Z}^m| \qquad\qquad \deg_\Sigma(h_P^m) = \mathrm{Vol}(P)$$

<u>Rem</u> $K(\Sigma) \simeq$ Grothendieck K-ring of vector bundles on X_Σ.

$H^*(\Sigma) \simeq$ cohomology ring of X_Σ.

Let $E = \{1, \cdots, n\}$

Let M be a <u>matroid</u> of rank r on E, and $\mathcal{B}$ its set of bases

(<u>linear matroid</u> : vectors $(v_1, \cdots, v_n)$ spanning $\mathbb{C}^r$, $\overset{\text{dual}}{\rightsquigarrow}$ $L \simeq \mathbb{C}^r \hookrightarrow \mathbb{C}^E$

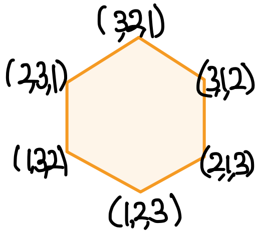
$\mathcal{B} = \{B \subseteq E \mid (v_i)_{i \in B} \text{ a linear basis of } \mathbb{C}^r\}.$)

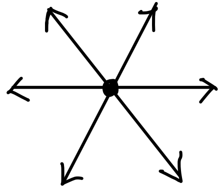
<u>E.g.</u> $L = \text{rowspan}\begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$ $\qquad \mathcal{B} = \{12, 13, 14, 23, 24\}$

$T_M(x,y)$ its Tutte polynomial.

$\Sigma = \Sigma_E$ the permutohedral fan

$= \sum \mathrm{Conv}\{\text{permutations of } (1,2,\cdots,n)\}$

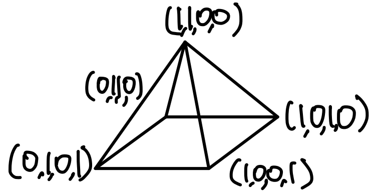




① Base polytope $P(M) = \mathrm{Conv}\{e_B : B \in \mathcal{B}\} \subset \mathbb{R}^E$

$\rightsquigarrow [P(M)] \in K(\Sigma_E)$

[Gelfand–Goresky–MacPherson–Serganova '87]



standard simplex

opposite simplex

<u>Thm</u> [Cameron–Fink '22]
[Benardi–Kalman–Postnikov '21]

$$\chi([P(M)][\Delta]^x[\nabla]^y)$$
$$\|$$
$$\#(P(M) + x\Delta + y\nabla) \text{ “}=\text{” } T_M(x,y)$$

② Bergman fan $\Sigma_M \rightsquigarrow [\Sigma_M] \in H^*(\Sigma_E)$ [Sturmfels '05] [Ardila-Klivans '06]

Thm [Huh-Katz '12] $\displaystyle\sum_{i=0}^{r-1} \deg_{\Sigma_E}\left([\Sigma_M]\cdot h_\Delta^{i} \cdot h_\nabla^{r-1-i}\right) q^i = T_M(q+1,0)/(q+1)$

Thm [Adiprasito-Huh-Katz '18][Ardila-Denham-Huh '23] Tropical Hodge thry

coeff.s of $T_M(q+1,0)$ and $T_M(q+1,1)$
form a log-concave sequence. ($c_i^2 \geq c_{i-1}c_{i+1}$).
(conj. by Heron, Rota, Welsh, Mason, ... in 70's)

$\boxed{Q1}$ $\chi([P(M)][\Delta]^x[\nabla]^y)$ "=" $T_M(x,y)$

$$\left.\begin{array}{l} \chi([P(M)][\Delta]^x[\nabla]^y) \text{ "=" } T_M(x,y) \\ \sum_{i=0}^{r-1} \deg_{\Sigma_E}\left([\Sigma_M]\cdot h_\Delta^{i}\cdot h_\nabla^{r-1-i}\right) q^i = T_M(q+1,0)/(q+1) \end{array}\right\} \text{Related ?}$$

$\boxed{Q2}$ Log-conc. for the whole T_M ?

$$[P] \longmapsto 1 + h_P + \frac{h_P^2}{2!} + \cdots$$

$$\begin{tikzcd} \dfrac{\mathbb{Z}\{[P] : P \text{ a def. of } \Sigma\}}{\langle [P]-[P+v] : v\in\mathbb{Z}^m\rangle} = K(\Sigma)_{\mathbb{Q}} \arrow[r, "\exp", "\simeq"'] \arrow[d, "\chi"'] \arrow[dr, phantom, "\text{HRR}"] & H^*(\Sigma)_{\mathbb{Q}} = \dfrac{\{\text{piecewise polynom. on } \Sigma\}}{\langle(\text{globally}) \text{ linear fcts}\rangle} \arrow[d, "\deg_\Sigma(-\,\cdot\,\mathrm{Td}(\Sigma))"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}$$

$[P][P'] = [P+P']$

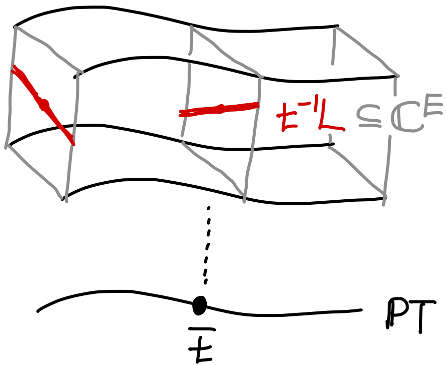
Let $T = (\mathbb{C}^*)^E$, acting on $\mathbb{C}^E$ $\quad t \cdot v = (t_1 v_1, \ldots, t_n v_n)$.

$\downarrow$

$\mathbb{P}T \ni \bar{t} = $ image of $t \in T$.

Defn Given r-dim'l $L \subseteq \mathbb{C}^E$ (i.e. $\mathbb{C}^E \twoheadrightarrow L^\vee \simeq \mathbb{C}^r$ realizes M),

define vec. subbundle $\mathcal{S}_L$ of $\underline{\mathbb{C}}^E = X \times \mathbb{C}^E$ $\left(\mathcal{Q}_L = \underline{\mathbb{C}}^E / \mathcal{S}_L \right)$ via:



$\mathbb{P}T \xhookrightarrow{\text{compactifies to}} X_E = $ permutohedral var.

Prop $[\mathcal{S}_L], [\mathcal{Q}_L] \in K(\Sigma)$ depend only on M.

Can define $[\mathcal{S}_M], [\mathcal{Q}_M] \in K(\Sigma)$, called tautological classes of M.

Thm [Berget–E–Spink–Tseng '23]

For $\mathcal{E} \in K(\Sigma_E)$, let $c_i(\mathcal{E}) \in H^{2i}(\Sigma_E)$ be its i-th Chern class.

① $c_1(Q_M) = h_{P(M)}$ and $c_{n-r}(Q_M) = [\Sigma_M]$

② $$\sum_{i,j,k,l} \deg_{\Sigma_E}\left(h_\Delta^i h_\nabla^j c_k(S_M) c_l(Q_M)\right) x^i y^j (-z)^k w^l = (x+y)^{-1}(y+z)^r (x+w)^{n-r} T_M\left(\frac{x+y}{y+z}, \frac{x+y}{x+w}\right)$$

③ Above polynom. is "array log-concave"

④ $\exists$ isom $\zeta : K(\Sigma_E) \xrightarrow{\simeq} H^*(\Sigma_E)$ such that

$$\chi(\mathcal{E}) = \deg_{\Sigma_E}\left(\zeta(\mathcal{E}) \cdot (1 + h_\Delta + \cdots + h_\Delta^{n-1})\right)$$

Recovers & extends: [Huh–Katz '12][Fink–Speyer '12][de Medrano–Rincon–Shaw '20]
[Bernardi–Kalman–Postnikov '21][Cameron–Fink '22]
[Adiprasito–Huh–Katz '18][Ardila–Denham–Huh '23]

Has led to:	
[E.–Huh–Larson]	augmented
[Ranganathan–Usatine '22]	Gromov–Witten
[Larson–Li–Payne–Proudfoot]	K-thry
[E.–Fink–Larson–Spink]	delta-matroids (type B)
[Ardila–E.–Penaguiao]	trop. crit.
[E.–Larson]	polymatroids
[E.]	sheaf cohom.
⋮	
?	