# The tropical critical points of an affine matroid

## FPSAC 2023, UC Davis


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MPI MiS Leipzig


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Slides can be found at `raulpenaguiao.github.io/`

Joint work with Federico Ardila and Christopher Eur

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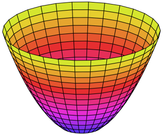

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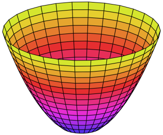

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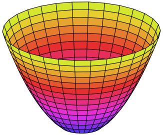

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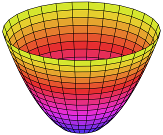

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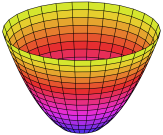

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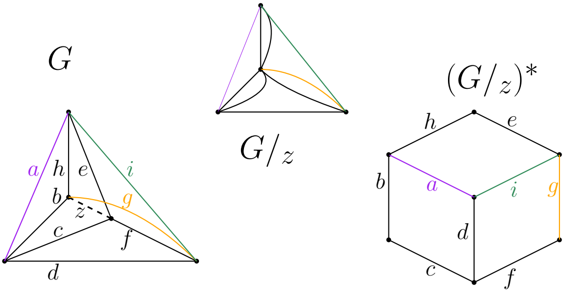
If $X$ is a vector space, MLDeg$(X) = \beta(M(V))$.

# Edge weight problem

Given $G = (V, E)$, fix some vector $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{Z}^n_{\geq 0}$.

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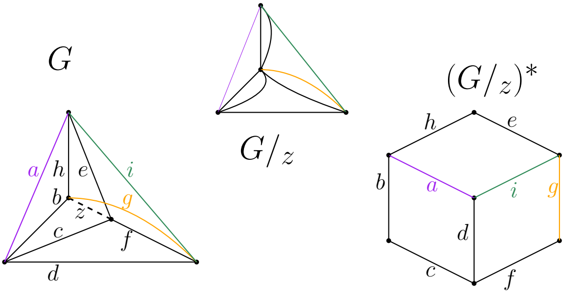


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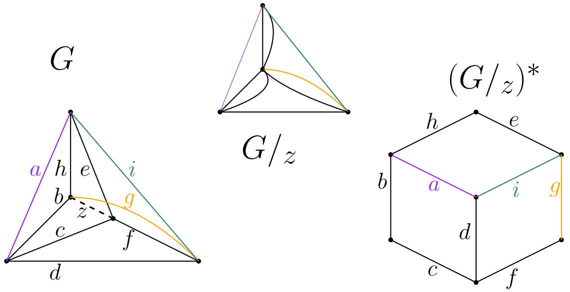


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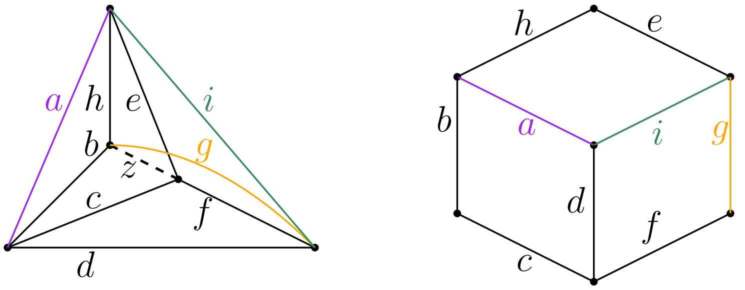
- The sum of the weights is $\mathbf{w}$.
- **(Compatible)** Every cycle has at least two minimal edges.

Fix $\mathbf{w} = (0, 1, 1, 2, 2, 5, 3, 4, 7)$.

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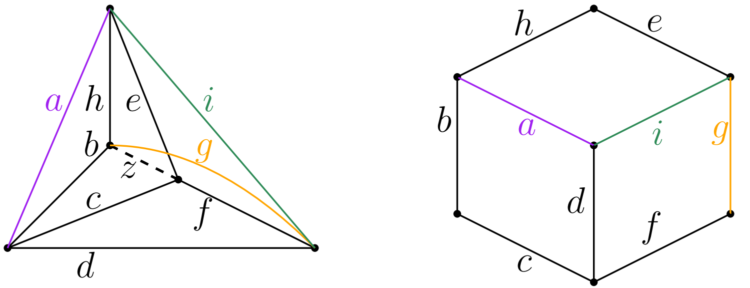


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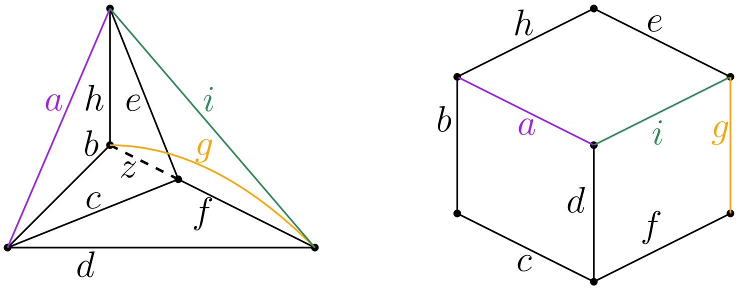


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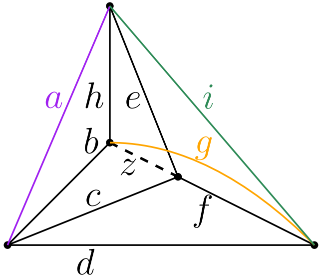


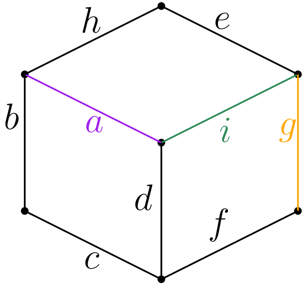


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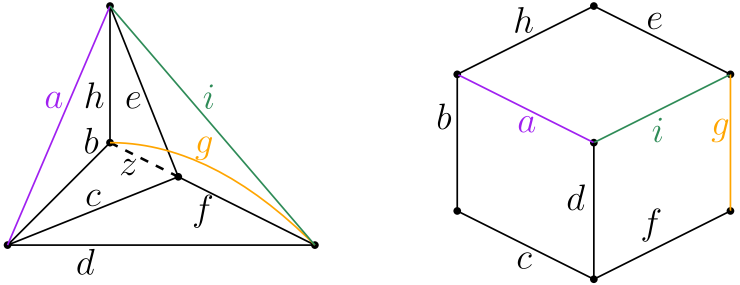


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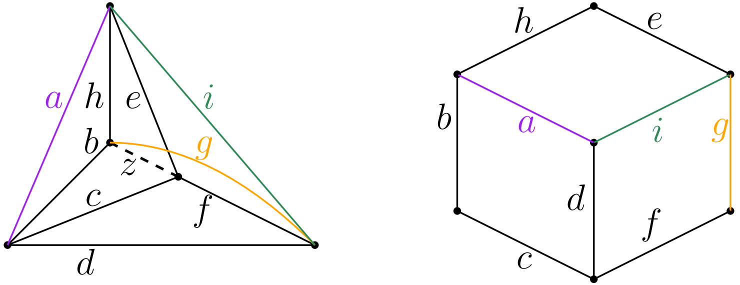


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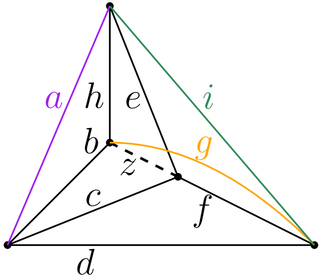


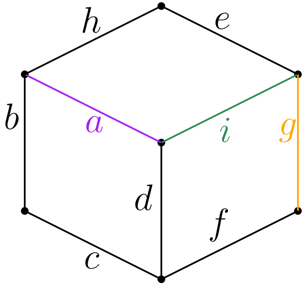


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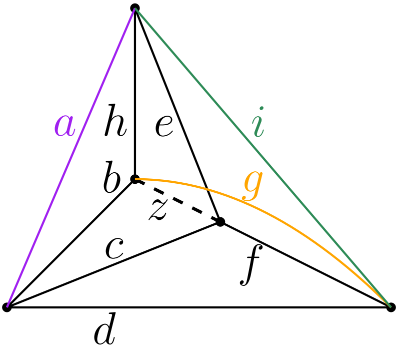
Given a graph $G = (V, E)$, the collection of edges $E$ forms a matroid.



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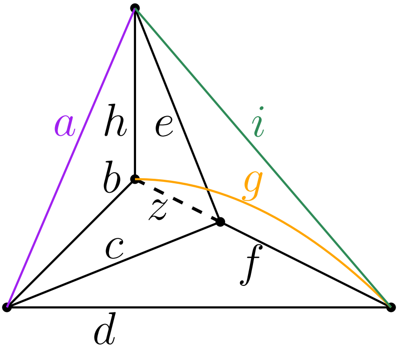


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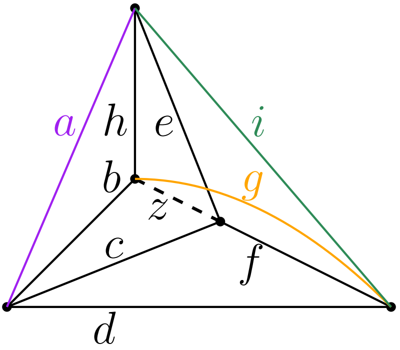


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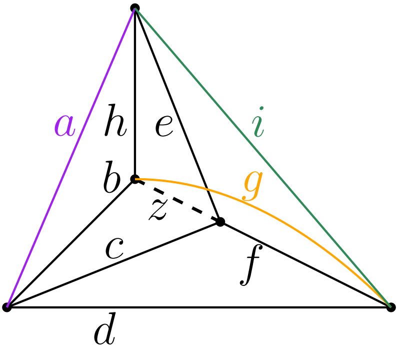


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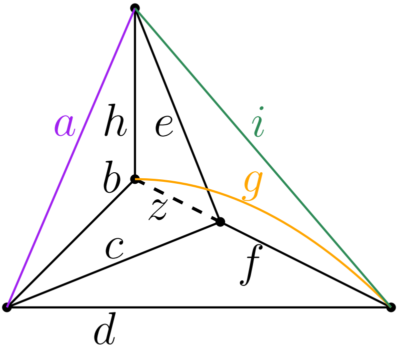


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Rank of set $A \subseteq E \mapsto$ size of largest spanning forest

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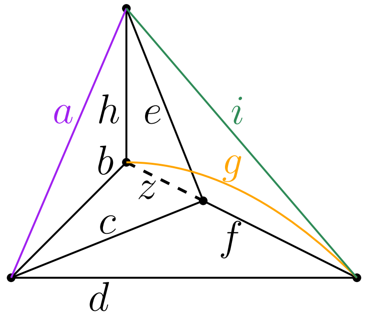
Maximal sets with a fixed rank.
That is, $F$ is a flat if for any $i \notin F$, $r_M(F \cup i) > r_M(F)$.



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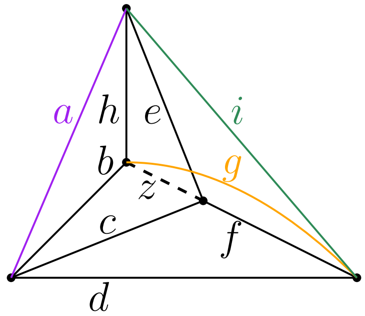


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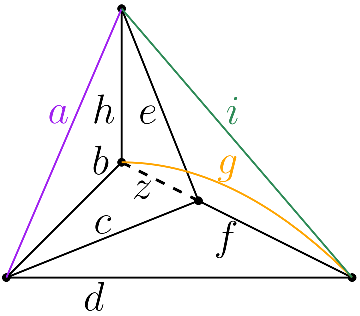


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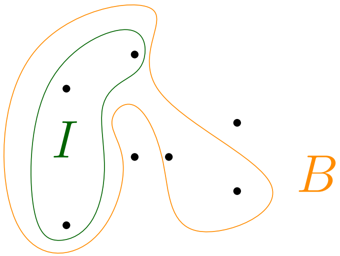
$$\{\emptyset \subsetneq a \subsetneq za \subsetneq zafg \subsetneq zabcdefghi\}$$

# The uniform matroid

Basis of the uniform matroid $U_{n,k}$ = all sets of size $k$ in $[n]$.
Any set of size $\leq k$ is independent.

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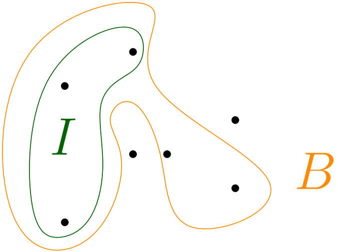


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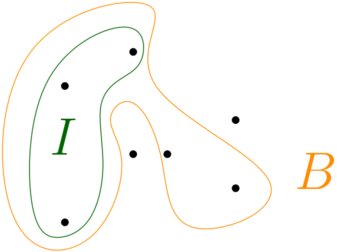


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Any complete flag of flats is of the form

$$\{\emptyset \subsetneq \{v_1\} \subsetneq \{v_1, v_2\} \subsetneq \cdots \subsetneq \{v_1, \ldots, v_{k-1}\} \subsetneq [n]\}$$

# The Bergman Fan

$$\Sigma_M := \{\vec{x} \in \mathbb{R}^n | \forall \text{ circuits } C \text{ we have } \min_{c \in C} x_c \text{ is attained twice } \} \subseteq \mathbb{R}^n /_{\mathbb{1}\mathbb{R}} .$$

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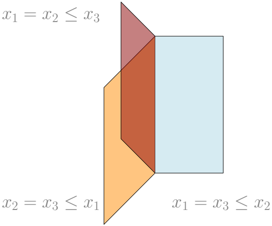


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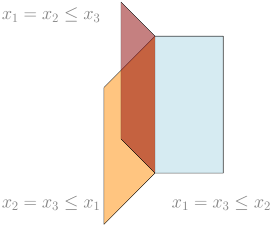


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$$\Sigma_{U_{n,k}} = \bigcup_{|I|=n-k+1} \{\vec{x} \in \mathbb{R}^n/_{\mathbb{R}\mathbb{1}} | \arg\min \vec{x} \subseteq I\} .$$

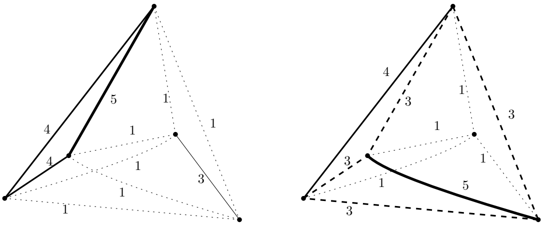
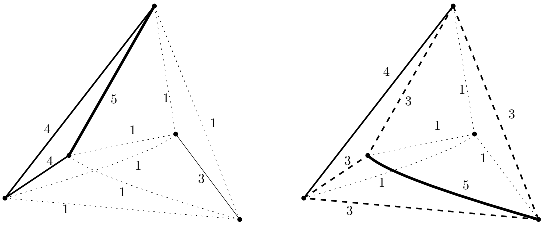



Figure: Two elements in the Bergman fan of the graphical matroid

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**Theorem (Sturmfels and Feichner, 2004)**

*The Bergman Fan of a matroid decomposes into the following cones*

$$\Sigma_M = \bigcup_{\mathcal{F} \text{ flag of flats}} \mathcal{C}_{\mathcal{F}} = \bigcup_{\substack{F_1 \subset \cdots \subset F_k \\ \text{flag of flats}}} \{x_i \geq x_j \text{ whenever } i \in F_k, j \notin F_k\} .$$

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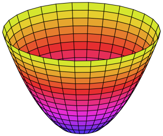

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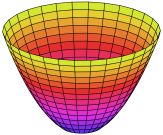

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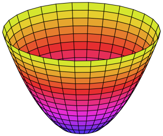

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Just intersect it with a hyperplane with dimension $= n - \dim X$
Note: Hyperplanes in the tropical world are Bergman fans of $U_{n,k}$.

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Consider $M$ the graphical matroid of $K_5$, of rank $4$ (so $\dim \Sigma_M = 3$).

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$$|\Sigma_{U_{10,7}} + \underbrace{(1, 10, 100, 1000, ..., 10^9)}_{\vec{\omega}} \cap \Sigma_M| = ?$$

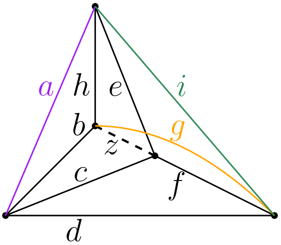


Figure: There is only one $\mathbf{x} \in \Sigma_M$ and $\mathbf{y} \in \Sigma_U$ such that $\mathbf{y} + \mathbf{w} = \mathbf{x}$. Such vector $\mathbf{x}$ belongs to the cone corresponding to the **greedy flag** of $M$.

# Activities

Fix total order in $V$, ground set of a matroid $M$, basis $B$.

- $e \in B$ is internal activity if $e = \min C^{\perp}$, where $C^{\perp} \subseteq B^c \cup e$ is a cocircuit (a cut of the graph).
- $e \notin B$ is external activity if $e = \min C$, where $C \subseteq B \cup i$ is a circuit.

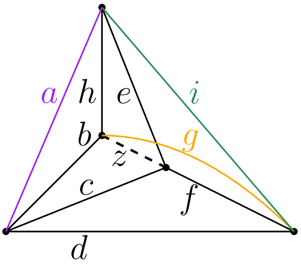


Figure: Fix order $z < a < b < c < d < e < f < g < h < i$.

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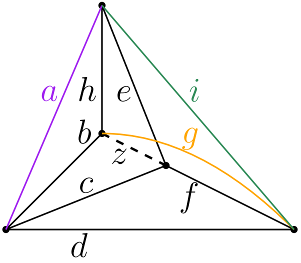


Figure: Fix order $z < a < b < c < d < e < f < g < h < i$.

$$i(befg) = 0,\ e(befg) = 2,\ i(zadf) = 2,\ e(zadf) = 1$$

$i(B) = 0$ nbc basis. $i(B) = 0$ and $e(B) = 1$ $\beta$-nbc basis

# Degree computations

**Theorem (Greedy basis algorithm)**

$$\deg(\Sigma_M) = \Sigma_M \cap (\mathbf{w} + \Sigma_{U_{n,n-k-1}}) = 1$$

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$$\deg(\Sigma_{(M/_0)^\perp} \cdot -\Sigma_M) = (-\Sigma_M) \cap (\mathbf{w} + \Sigma_{(M/_0)^\perp}) = \#\{\beta - \textit{nbc bases}\}$$

# Finding ~~Nemo~~ points form nbc bases

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$\deg(\Sigma_{(M/_0)^\perp} \cdot -\Sigma_M) =?$ Fix generic $\mathbf{w} = (10^{i-1})_i$.
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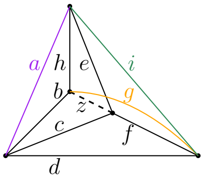


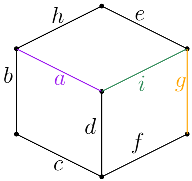


Find a $\beta$-nbc basis $B = zbeg \to g|e|bd|zacfhi$,

corresponding $\beta$-nbc cobasis $B^\perp = acdfhi \to i|h|f|gd|c|eba$.

| | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | $h$ | $i$ |
|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{x}$ | | | | | | | | | |
| $\mathbf{y}$ | | | | | | | | | |
| $\mathbf{w}$ | $1$ | $10$ | $100$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ | $10^7$ | $10^8$ |

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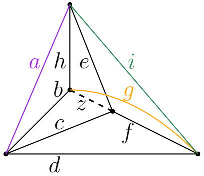


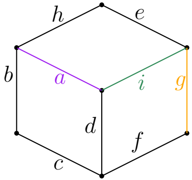


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| | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | $h$ | $i$ |
|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{x}$ | 0 | | 0 | | | 0 | | 0 | 0 |
| $\mathbf{y}$ | | | | | | | | | |
| $\mathbf{w}$ | 1 | 10 | 100 | $10^3$ | $10^4$ | $10^5$ | $10^6$ | $10^7$ | $10^8$ |

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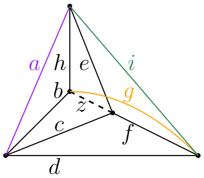


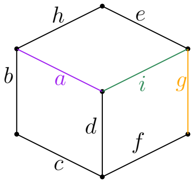


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|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{x}$ | 0 | | 0 | | | 0 | | 0 | 0 |
| $\mathbf{y}$ | 1 | | 100 | | | $10^5$ | | $10^7$ | $10^8$ |
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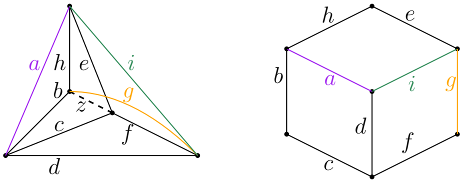


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|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{x}$ | 0 | | 0 | | | 0 | | 0 | 0 |
| $\mathbf{y}$ | 1 | 1 | 100 | | 1 | $10^5$ | | $10^7$ | $10^8$ |
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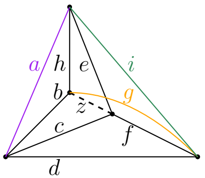


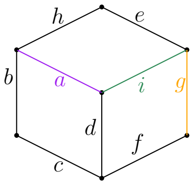


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| | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | $h$ | $i$ |
|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{x}$ | 0 | | 0 | | $10^4 - 1$ | 0 | | 0 | 0 |
| $\mathbf{y}$ | 1 | 1 | 100 | | 1 | $10^5$ | | $10^7$ | $10^8$ |
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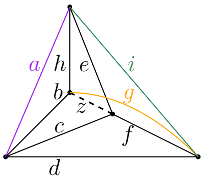


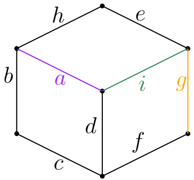


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| | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | $h$ | $i$ |
|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{x}$ | 0 | | 0 | 9 | $10^4 - 1$ | 0 | | 0 | 0 |
| $\mathbf{y}$ | 1 | 1 | 100 | | 1 | $10^5$ | | $10^7$ | $10^8$ |
| $\mathbf{w}$ | 1 | 10 | 100 | $10^3$ | $10^4$ | $10^5$ | $10^6$ | $10^7$ | $10^8$ |

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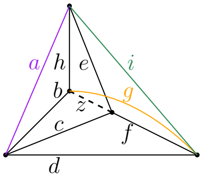


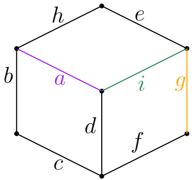


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|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{x}$ | 0 | 9 | 0 | 9 | $10^4 - 1$ | 0 | | 0 | 0 |
| $\mathbf{y}$ | 1 | 1 | 100 | $10^3 - 9$ | 1 | $10^5$ | | $10^7$ | $10^8$ |
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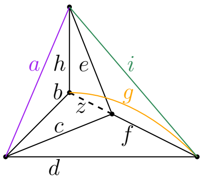


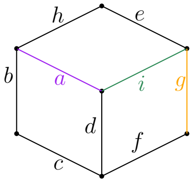


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| $\mathbf{x}$ | 0 | 9 | 0 | 9 | $10^4 - 1$ | 0 | | 0 | 0 |
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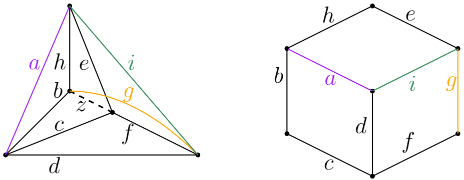


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| | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | $h$ | $i$ |
|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{x}$ | 0 | 9 | 0 | 9 | $10^4 - 1$ | 0 | $10^6 - 10^3 + 9$ | 0 | 0 |
| $\mathbf{y}$ | 1 | 1 | 100 | $10^3 - 9$ | 1 | $10^5$ | $10^3 - 9$ | $10^7$ | $10^8$ |
| $\mathbf{w}$ | 1 | 10 | 100 | $10^3$ | $10^4$ | $10^5$ | $10^6$ | $10^7$ | $10^8$ |

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# Thank you