Pizza and 2-structures

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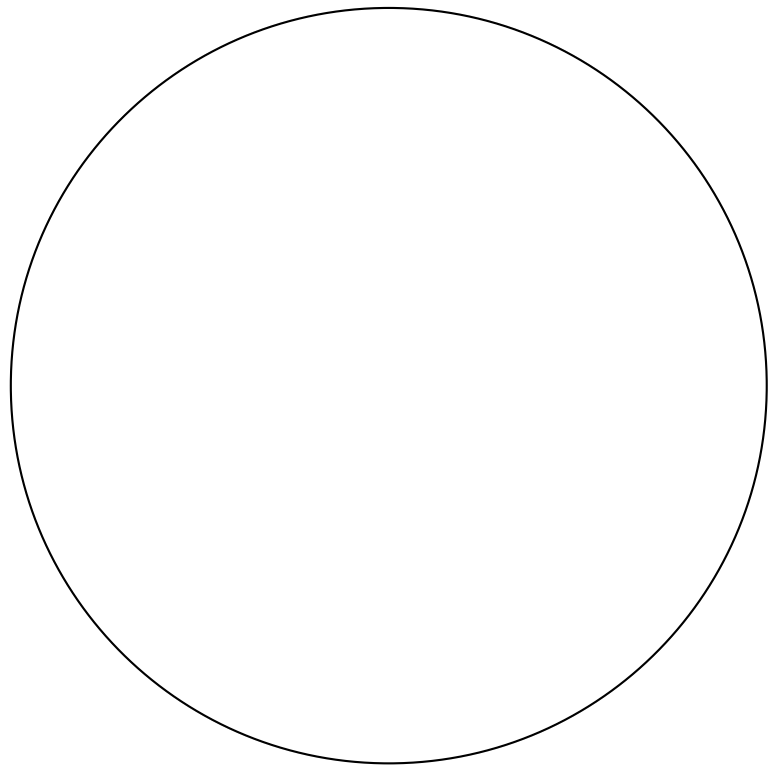
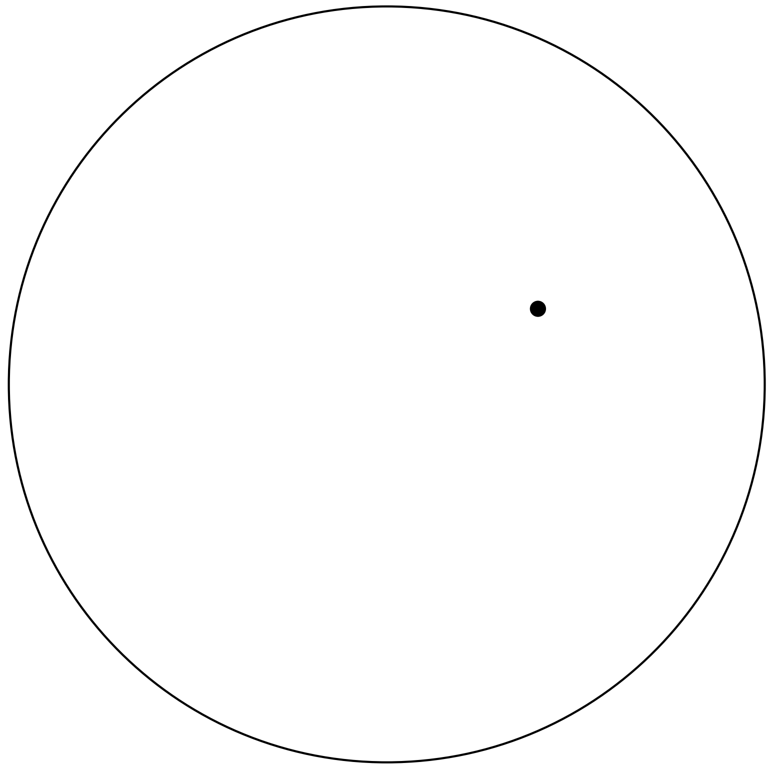
Pizza

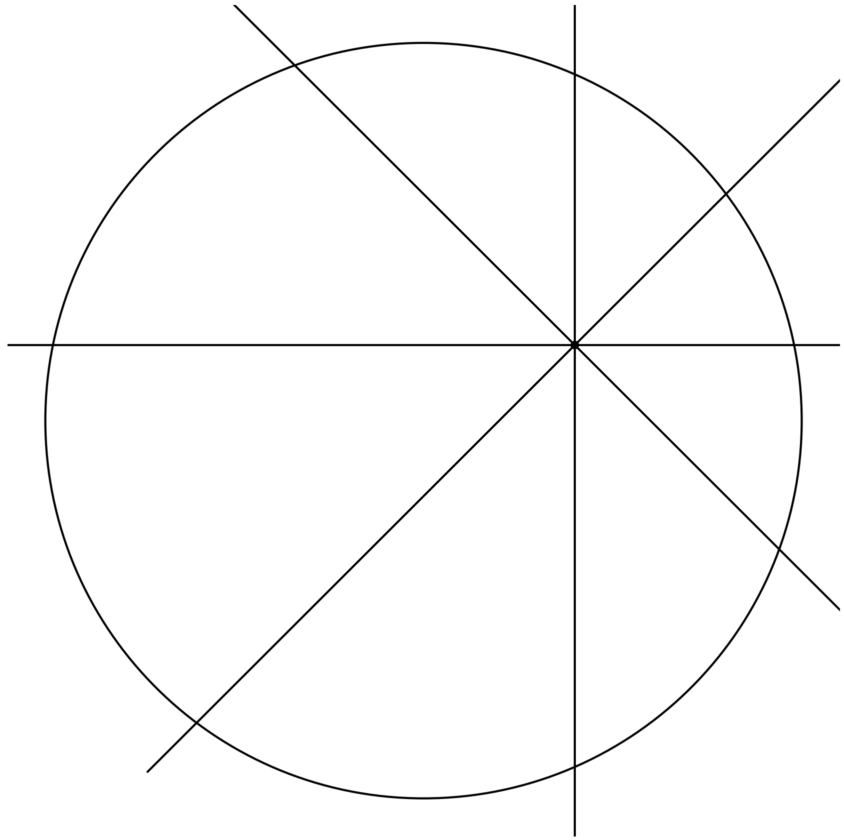
Pizza



Pick any point

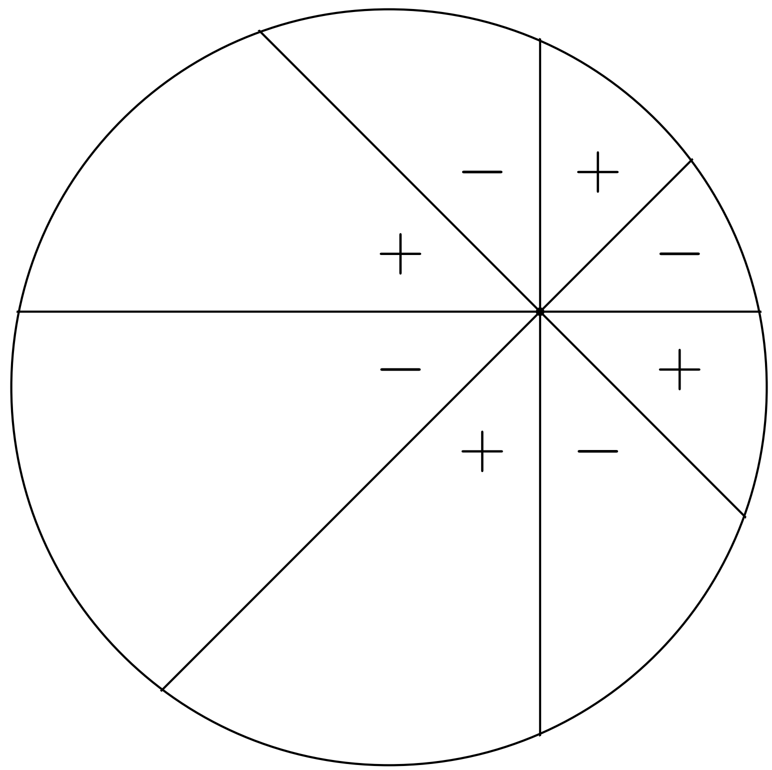


Cut with four equidistributed lines



Pizza Theorem [Goldberg]

The alternating sum of the areas is equal to 0.



History

[1967, Upton] Problem in *Mathematics Magazine*.

[1968, Goldberg] Solution for $2k$ equidistributed lines $k \geq 2$.

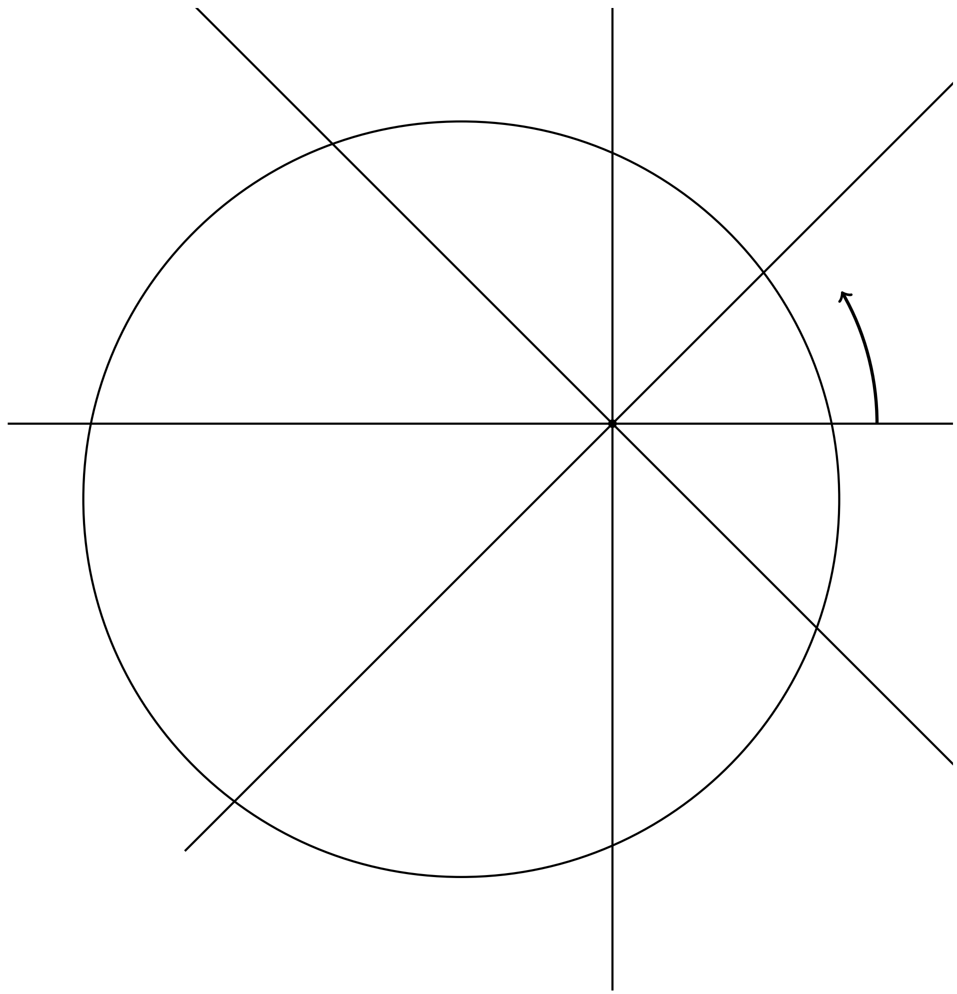
[1994, Carter and Wagon] Dissection proof for $k = 2$.

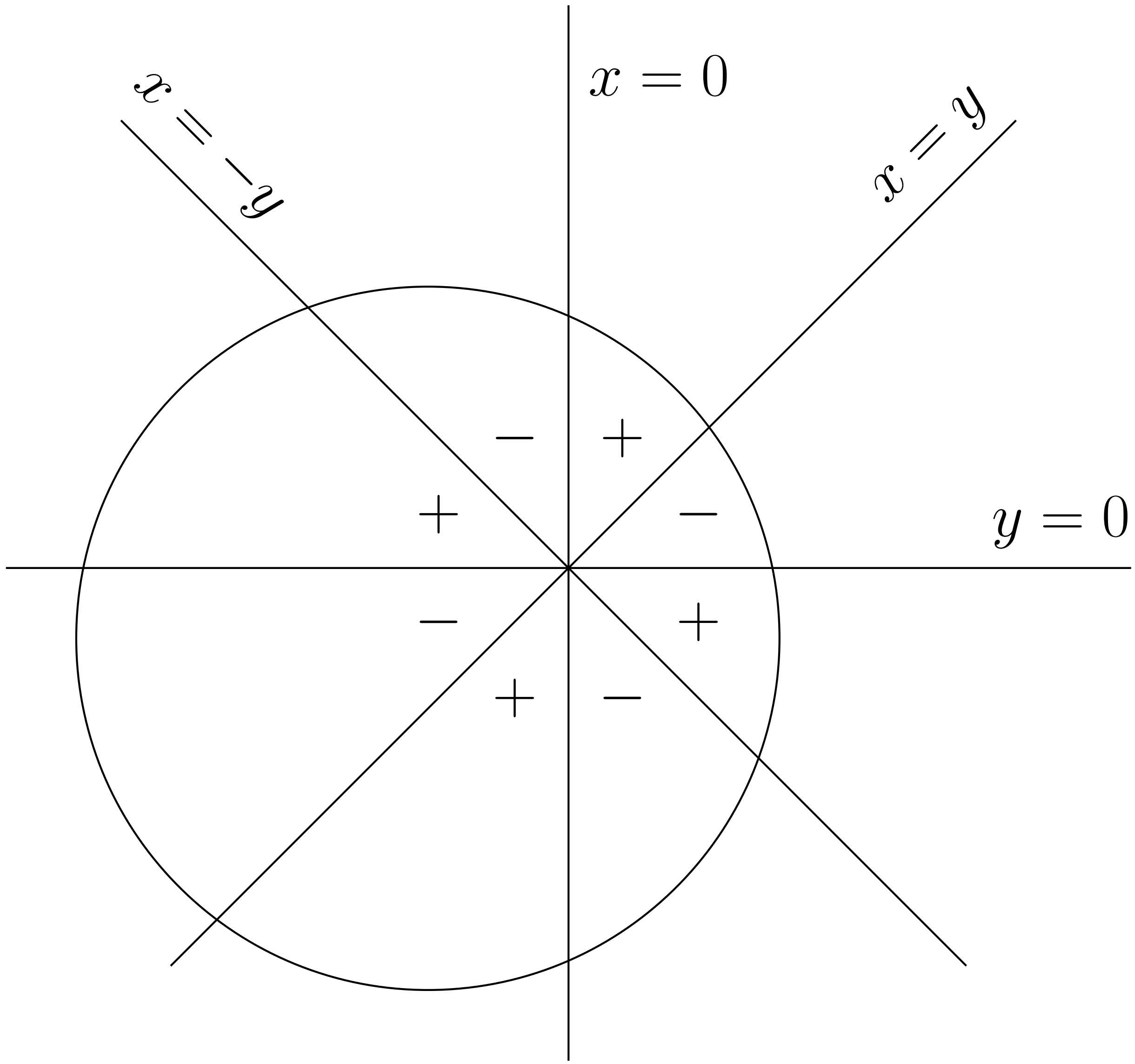
[1999, Hirschhorn, Hirschhorn, Hirschhorn, Hirschhorn and Hirschhorn] p people sharing pizza.

[2009, Mabry and Deiermann] Fails for an odd number of equidistributed lines.

[2012, Frederickson] Dissection proofs for $k \geq 2$.

Classical proof





$x = 0$
$x = -y$
$x = y$
$y = 0$
$-$
$+$
$+$
$-$
$-$
$+$
$+$
$-$

B_2 or not B_2: that is the question

William Shakespeare, Hamlet, Act III

V real vector space of dimension n with inner product $(\cdot, \cdot)$

Index set E finite set of unit vectors such that $E \cap (-E) = \emptyset$

Hyperplane $H_e = \{v \in V : (v, e) = 0\}$

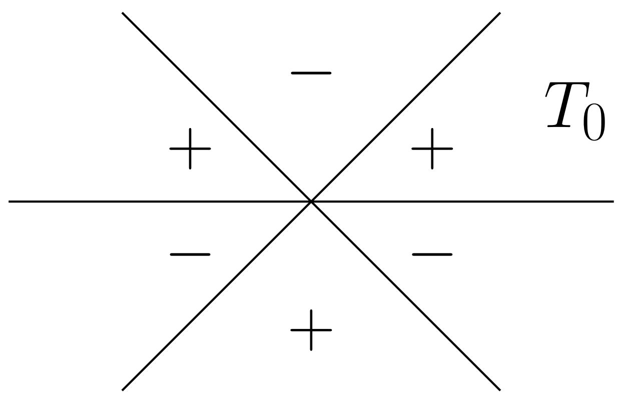
Hyperplane arrangement $\mathcal{H} = \{H_e\}_{e \in E}$

A chamber T is a connected component of $V - \bigcup_{e \in E} H_e$

$\mathscr{T}$ set of all chambers

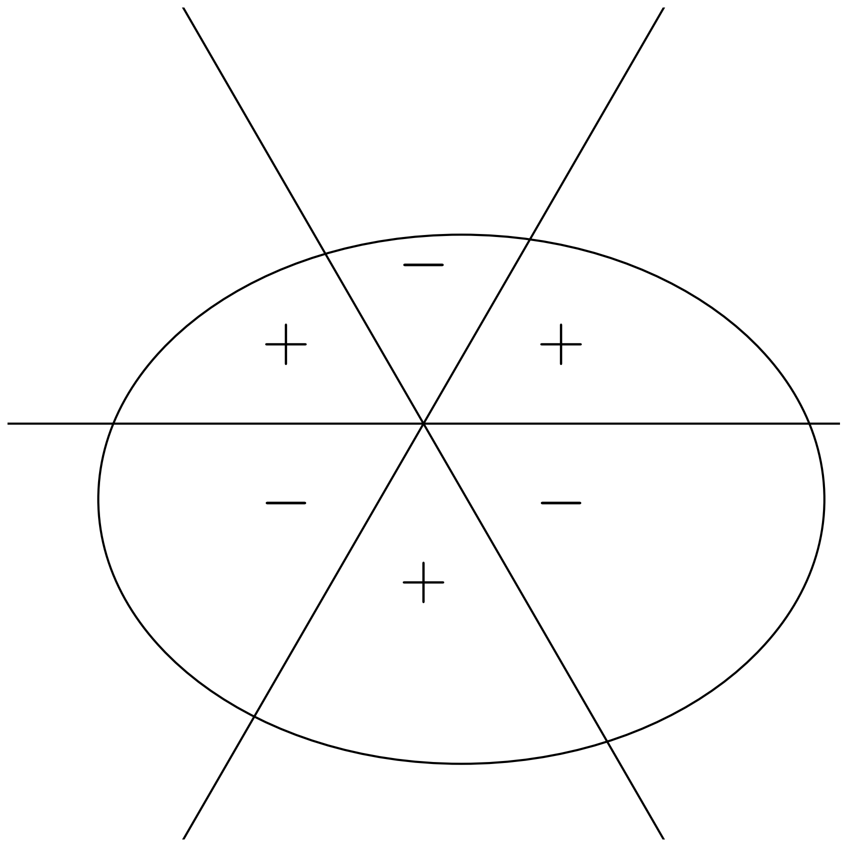
Pick T_0 base chamber

Sign $(-1)^T = (-1)^k$ where k is the number of hyperplanes separating T from T_0



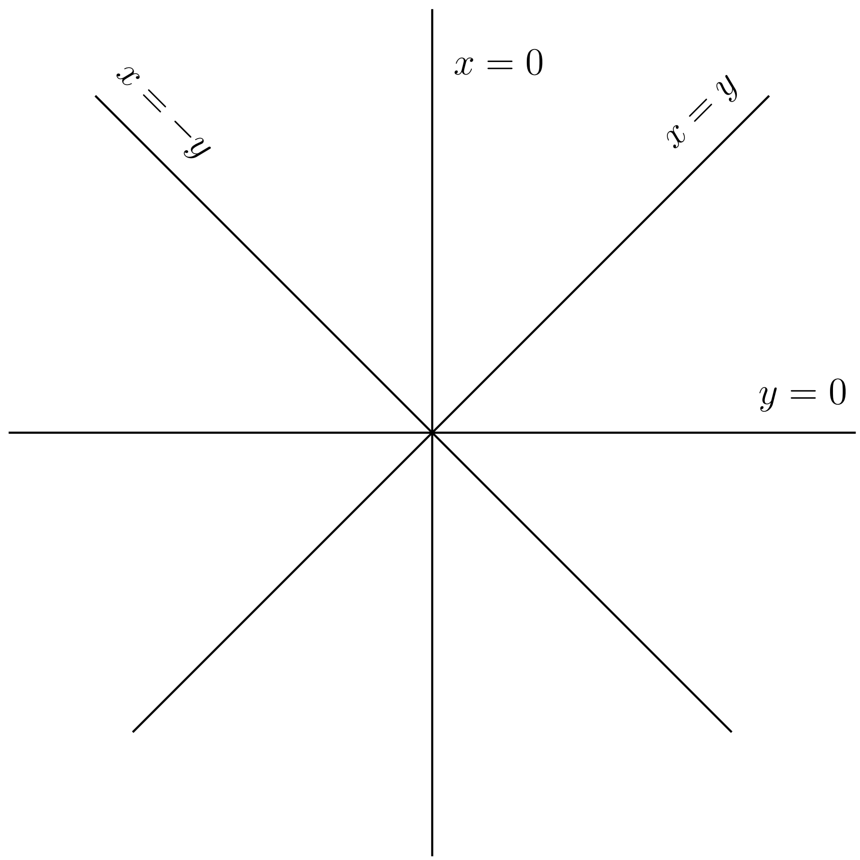
Pizza quantity

$$P(\mathcal{H}, K) = \sum_{T \in \mathscr{T}} (-1)^{T} \operatorname{Vol}(K \cap T)$$



$\mathcal{H}$ is a *Coxeter arrangement* if

- the group W generated by the orthogonal reflections in the hyperplanes of $\mathcal{H}$ is finite and
- the arrangement is closed under all such reflections



$\mathcal{H}_i$ arrangement in V_i

$\mathcal{H}_1 \times \mathcal{H}_2$ arrangement in $V_1 \times V_2$ with hyperplanes

$$\{H \times V_2 : H \in \mathcal{H}_1\} \cup \{V_1 \times H : H \in \mathcal{H}_2\}$$

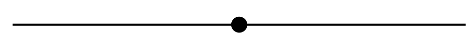
$\mathcal{H}_1$ and $\mathcal{H}_2$ Coxeter $\Longrightarrow \mathcal{H}_1 \times \mathcal{H}_2$ Coxeter

Type A_n

$$V = \{(x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1 + x_2 + \cdots + x_{n+1} = 0\}$$

$$\mathcal{H} = \{x_i = x_j : 1 \le i < j \le n+1\}$$

Symmetries of the n-dimensional simplex

A_1 

$$\begin{aligned} A_1^n &= A_1 \times A_1 \times \cdots \times A_1 \\ &= \{x_i = 0 : 1 \le i \le n\} \end{aligned}$$

Type B_n (and type C_n) $\quad n \geq 2$

$$V = \mathbb{R}^n$$

$$\mathcal{H} = \{x_i = 0 : 1 \leq i \leq n\} \cup \{x_i = \pm x_j : 1 \leq i < j \leq n\}$$

Symmetries of the n-dimensional cube and crosspolytope

Type D_n $\quad n \geq 4$

$$V = \mathbb{R}^n$$

$$\mathcal{H} = \{x_i = \pm x_j : 1 \leq i < j \leq n\}$$

$D_2 = A_1^2 \qquad D_3 = A_3$

Type E_6, E_7 and E_8

Type F_4

$$V = \mathbb{R}^4$$

$$\begin{aligned}\mathcal{H} = &\{x_i = 0 : 1 \leq i \leq 4\} \\ &\cup \{x_i = \pm x_j : 1 \leq i < j \leq 4\} \\ &\cup \{x_1 \pm x_2 \pm x_3 \pm x_4 = 0\}\end{aligned}$$

F_4 = symmetries of the 24-cell

Type G_2

$G_2 = I_2(6)$

Type H_3 and H_4

$H_3 =$ symmetries of the dodecahedron and the icosahedron

$H_4 =$ symmetries of the 120-cell and 600-cell

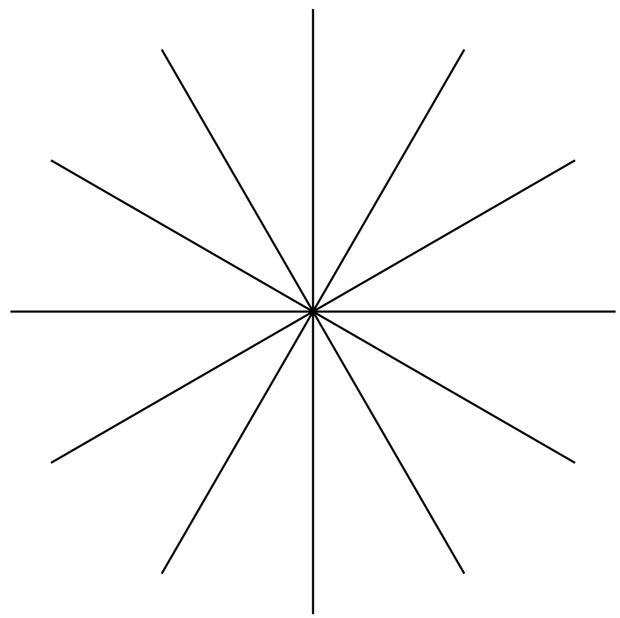
Do not arise from crystallographic root systems

Type $I_2(k)$ $\qquad$ $k \geq 2$

$I_2(k) =$ symmetries of the k-gon

$I_2(k)$ consists of k lines

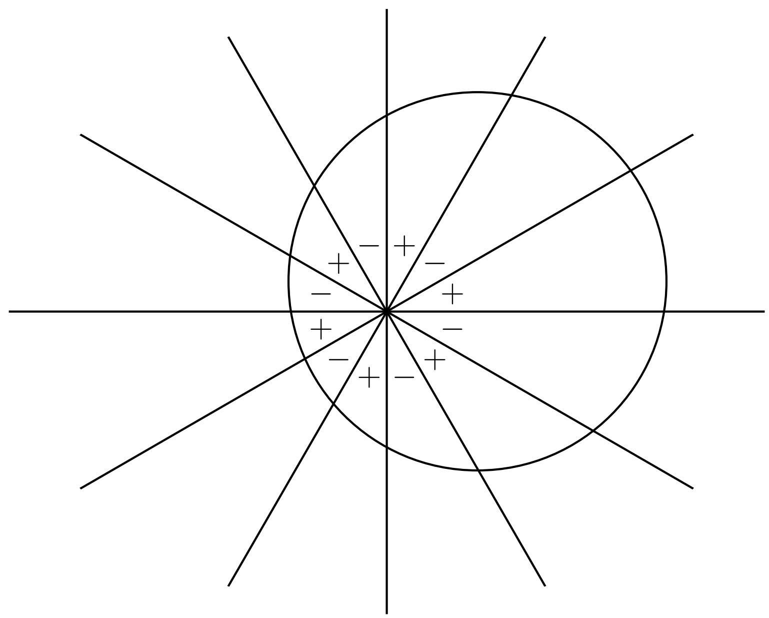
$I_2(2) = A_1^2$ $\qquad$ $I_2(3) = A_2$ $\qquad$ $I_2(4) = B_2$



$$\mathbb{B}(a, R) = \{x \in V : \|x - a\| \leq R\}.$$

Theorem [Goldberg] Let $\mathcal{H}$ be the dihedral arrangement $I_2(2k)$ in $\mathbb{R}^2$ for $k \geq 2$. For every point $a \in \mathbb{R}^2$ such that $0 \in \mathbb{B}(a, R)$, the pizza quantity for the disc $\mathbb{B}(a, R)$ vanishes:

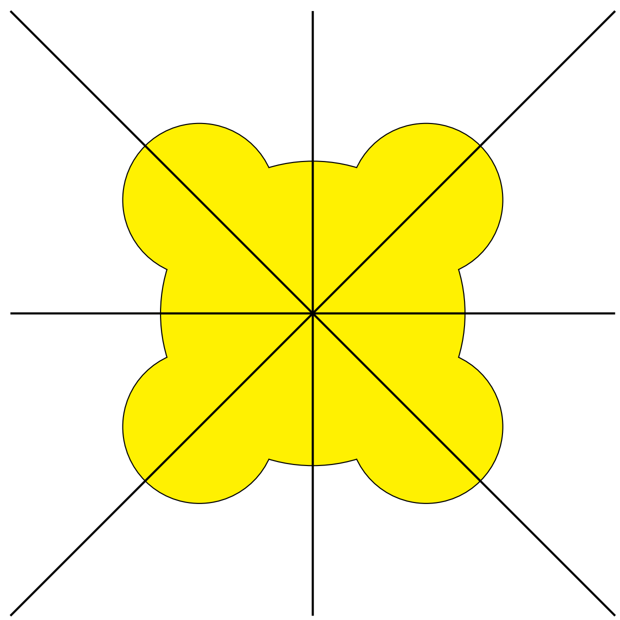
$$P(\mathcal{H}, \mathbb{B}(a, R)) = 0.$$



A set $K \subseteq V$ is stable under the group W if

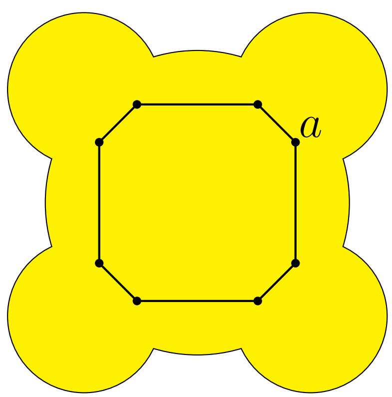
$$w(K) = K$$

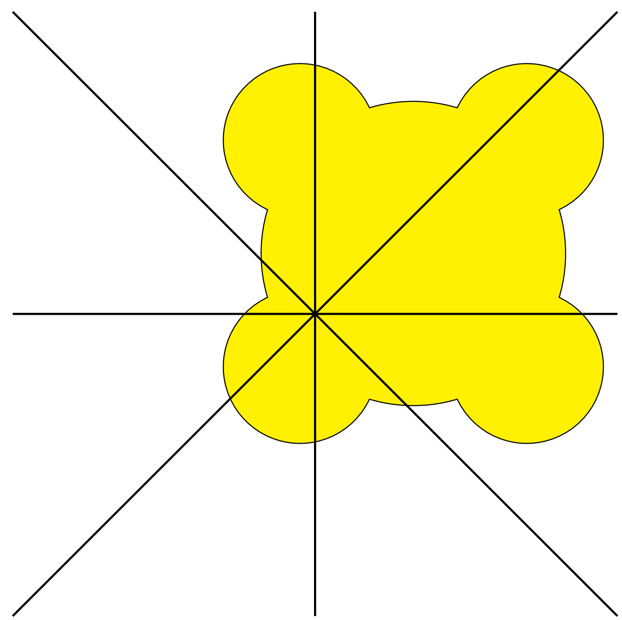
for all $w \in W$



Theorem. Let $\mathcal{H}$ be a Coxeter arrangement on V such that the negative of the identity map $-\mathrm{id}_V$ belongs to the Coxeter group W. Assume that $\mathcal{H}$ is not of type A_1^n. Let K be a set stable by W. Let a be a point in V such that K contains the convex hull of $\{w(a) : w \in W\}$. Then the pizza quantity of $K + a$ vanishes, that is,

$$P(\mathcal{H}, K + a) = 0.$$





History continued

[2012, Frederickson] Type $A_1 \times I_2(2k)$ for $k \geq 2$ for balls.

[2022, Brailov] Independently proved the theorem for type B_n for balls.

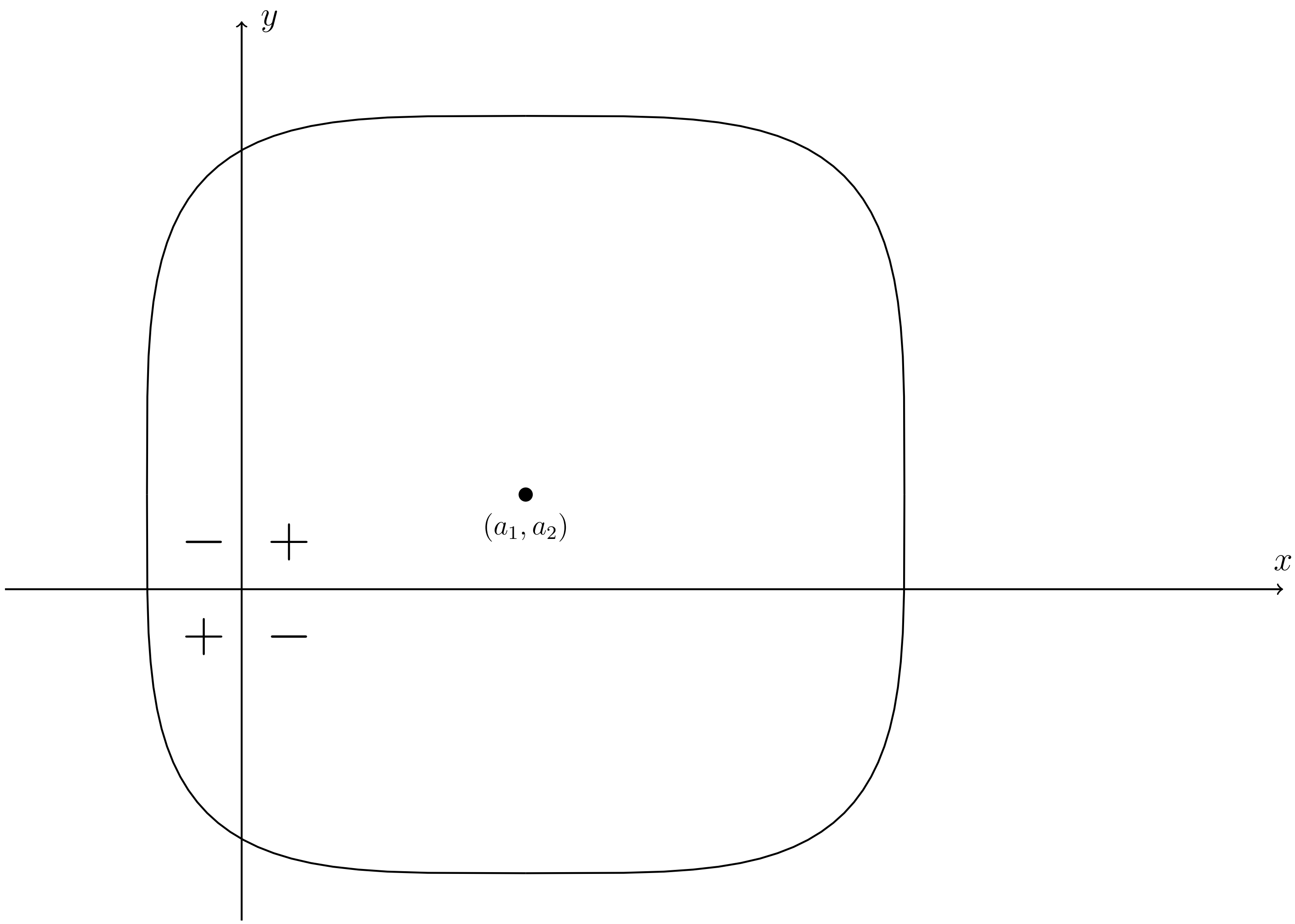
$$-\operatorname{id}_V \in W$$

$$\iff$$

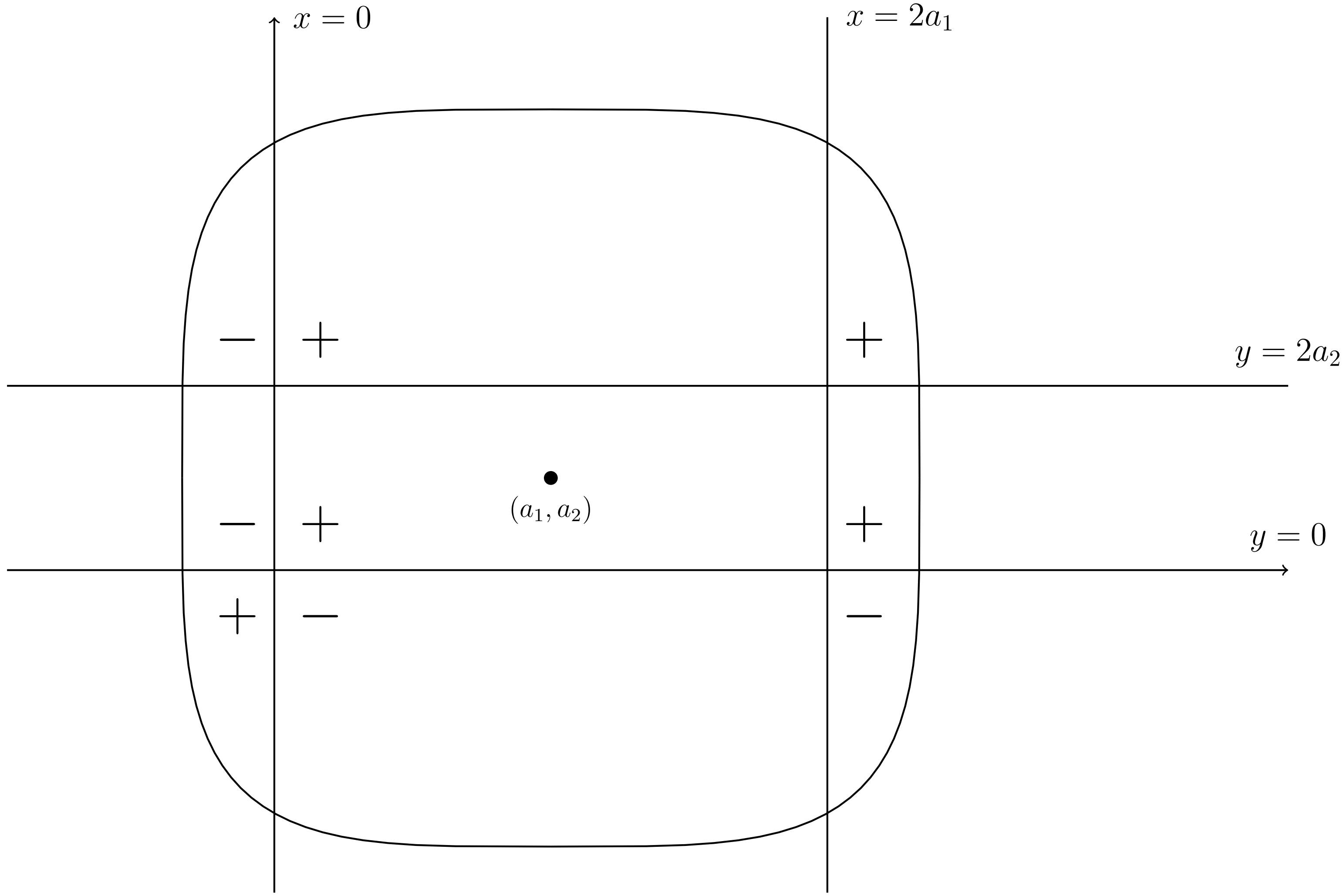
$\mathcal{H}$ is a product arrangement where the factors are from the types A_1, B_n for $n \geq 2$, D_{2m} for $m \geq 2$, E_7, E_8, F_4, H_3, H_4 and $I_2(2k)$ for $k \geq 2$.

Missing: A_n for $n \geq 2$,
D_{2m+1} for $m \geq 2$,
E_6,
$I_2(2k+1)$ for $k \geq 2$.

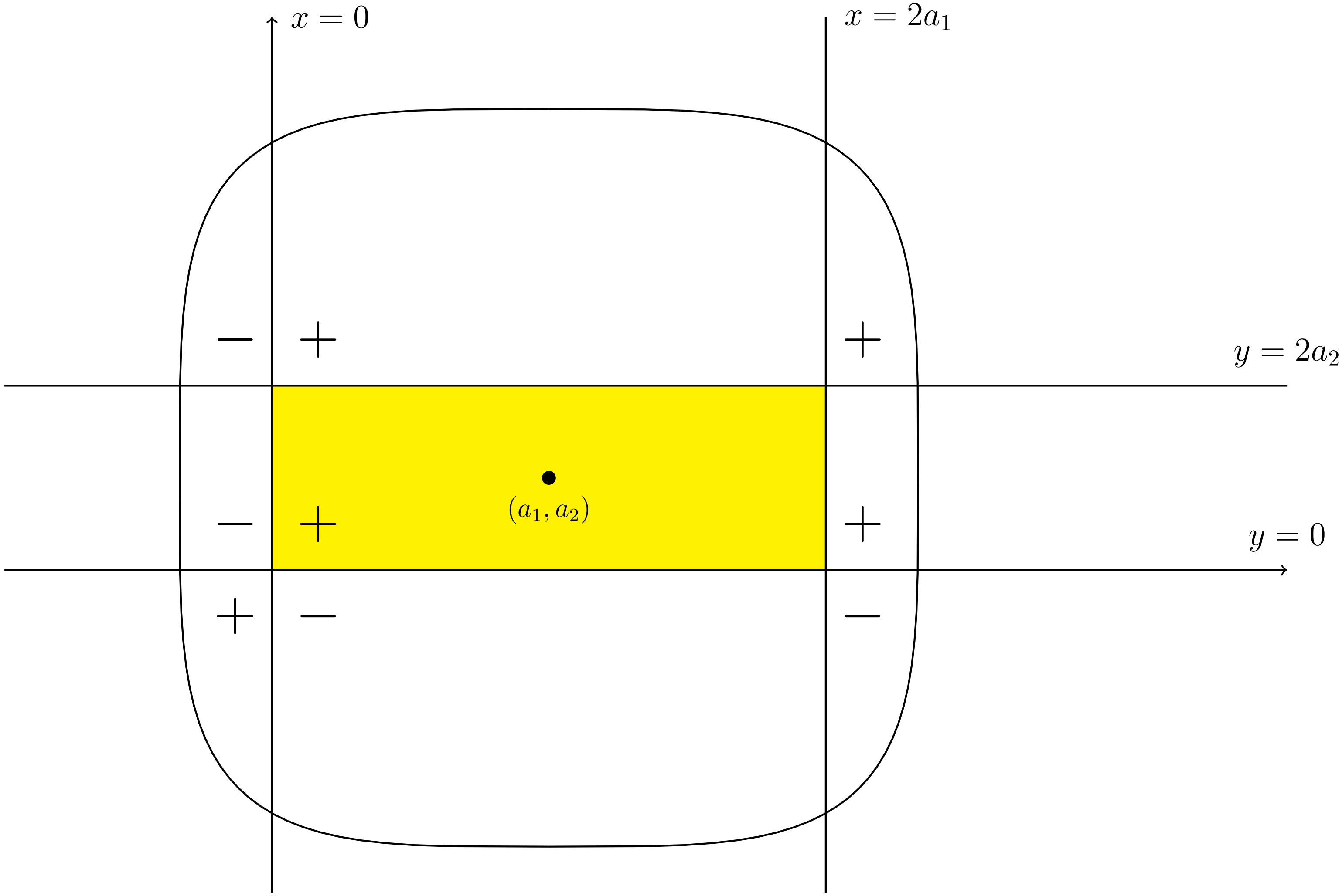
What happens with A_1^n?



Cut also with $x_i = 2a_i$.

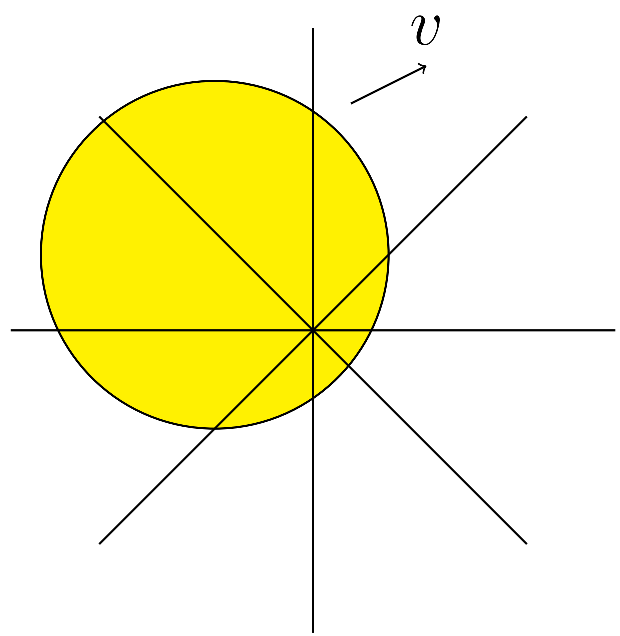


$$P(A_1^n, K + (a_1, \ldots, a_n)) = 2^n \cdot a_1 \cdots a_n$$



Proof:

$$\frac{d}{dt}P(\mathcal{H}, K + t \cdot v) =?$$



[Ira Gessel, October 28, 2006]

Is Analysis Necessary?

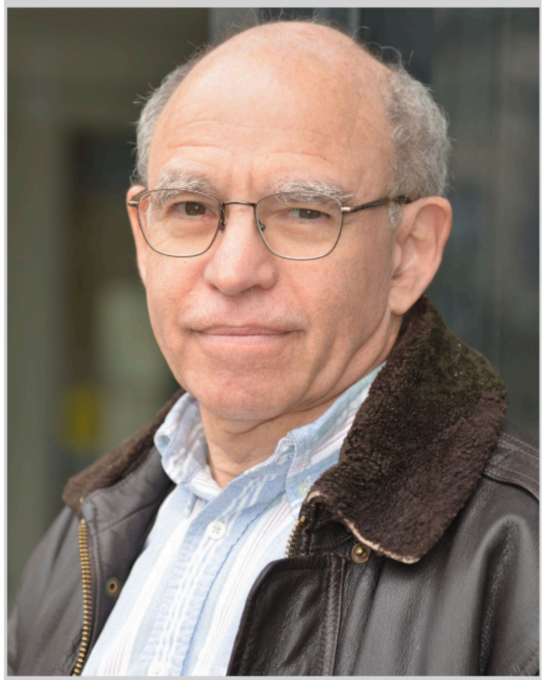


Photo by Michael Gessel

The best way to show that the two sets

$$\bigcup_{\substack{T \\ (-1)^T=1}} ((K+a)\cap T) \qquad \text{and} \qquad \bigcup_{\substack{T \\ (-1)^T=-1}} ((K+a)\cap T)$$

have the same volume, is a dissection proof.

Definition. Let $\mathcal{C}(V)$ be a *nice* family of subsets of V, satisfying:

(i) closed by finite intersections,

(ii) affine isometries,

(iii) if $C \in \mathcal{C}(V)$ and D is a closed affine half-space of V, then $C \cap D \in \mathcal{C}(V)$ and

(iv) closed with respect to Cartesian products, that is, if $C_i \in \mathcal{C}(V_i)$ for $i = 1, 2$ then $C_1 \times C_2 \in \mathcal{C}(V_1 \times V_2)$.

Definition. We denote by $K(V)$ the quotient of the free abelian group $\bigoplus_{C \in \mathcal{C}(V)} \mathbb{Z}[C]$ on $\mathcal{C}(V)$ by the relations:

– $[\varnothing] = 0$;

– $[C \cup C'] + [C \cap C'] = [C] + [C']$ for all $C, C' \in \mathcal{C}(V)$ such that $C \cup C' \in \mathcal{C}(V)$;

– $[g(C)] = [C]$ for $C \in \mathcal{C}(V)$ and affine isometry g of V.

For $C \in \mathcal{C}(V)$ we still denote the image of C in $K(V)$ by $[C]$.

K pizza

$\mathcal{H}$ hyperplane arrangement

Define the *abstract pizza quantity* to be

$$P(\mathcal{H}, K) = \sum_{T \in \mathscr{T}(\mathcal{H})} (-1)^{T} \cdot [T \cap K].$$

The Abstract Pizza Theorem.
Let $\mathcal{H}$ be a Coxeter hyperplane arrangement with Coxeter group W in an n-dimensional space V such that $-\mathrm{id}_V \in W$. Assume that $\mathcal{H}$ does not have type A_1^n. Let $K \in \mathcal{C}(V)$ and $a \in V$. Suppose that K is stable by the group W and contains the convex hull of the set $\{w(a) : w \in W\}$. Then the abstract pizza quantity vanishes:

$$P(\mathcal{H}, K + a) = 0,$$

that is, this identity holds in $K(V)$.

Let s_β be the orthogonal reflection in the hyperplane H_β.

Definition. A subset Φ of V is a *pseudo-root system* if:

(a) Φ is a finite set of unit vectors;
(b) for all $\alpha, \beta \in \Phi$, we have $s_\beta(\alpha) \in \Phi$.

Note that condition (b) implies that $\alpha \in \Phi$ implies $-\alpha \in \Phi$ by setting $\alpha = \beta$. Elements of Φ are called *pseudo-roots*.

$$\Phi = \Phi^+ \sqcup \Phi^-$$

Φ^+ = positive pseudo-roots,
Φ^- = negative pseudo-roots.

Definition [Herb]. Let Φ be a pseudo-root system with Coxeter group W. A 2-*structure* for Φ is a subset φ of Φ satisfying the following properties:

(a) The subset φ is a disjoint union
$$\varphi = \varphi_1 \sqcup \varphi_2 \sqcup \cdots \sqcup \varphi_r,$$
where the φ_i are pairwise orthogonal subsets of φ and each of them is an irreducible pseudo-root system of type A_1, B_2 or $I_2(2^k)$ for $k \geq 3$.

(b) Let $\varphi^+ = \varphi \cap \Phi^+$. If w is an element in W such that $w(\varphi^+) = \varphi^+$ then the sign of w is positive, that is, $(-1)^w = 1$.

History continued

[2000, Herb] Introduced 2-structures to study the characters of discrete series representations.

Let $\mathcal{T}(\Phi)$ denote the set of 2-structures for Φ.

The group W acts transitively on $\mathcal{T}(\Phi)$.

Hence all 2-structures of Φ have the same type.

Type of Φ	Type of φ	Type of Φ	Type of φ	
A_{2m}	A_1^m	E_7	A_1^7	
A_{2m+1}	A_1^{m+1}	E_8	A_1^8	
B_{2m}	B_2^m	F_4	B_2^2	
B_{2m+1}	$B_2^m \times A_1$	H_3	A_1^3	
D_{2m}	A_1^{2m}	H_4	A_1^4	
D_{2m+1}	A_1^{2m}	$I_2(r)$	A_1	$(r \text{ odd})$
E_6	A_1^4	$I_2(r \cdot 2^k)$	$I_2(2^k)$	$(k \geq 1)$

Φ pseudo-root system

φ 2-structure of Φ

$$\operatorname{rank}(\Phi) = \operatorname{rank}(\varphi) \qquad \Longleftrightarrow \qquad -\operatorname{id} \in W$$

Each 2-structure has a sign, that is,

$$\epsilon : \mathcal{T}(\Phi) \longrightarrow \{\pm 1\}.$$

Properties:

(i)

$$\sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon(\varphi) = 1$$

(ii) For $w \in W$ such that $w(\varphi \cap \Phi^+) \subseteq \Phi^+$. Then the following identity holds:

$$\epsilon(w(\varphi)) = (-1)^w \cdot \epsilon(\varphi).$$

Theorem. Let $\Phi \subset V$ be a normalized pseudo-root system. Choose a positive system $\Phi^+ \subset \Phi$ and let $\mathcal{H}$ be the hyperplane arrangement $(H_\alpha)_{\alpha \in \Phi^+}$ on V with base chamber corresponding to Φ^+. For every 2-structure $\varphi \in \mathcal{T}(\Phi)$, let $\mathcal{H}_\varphi$ be the hyperplane arrangement $(H_\alpha)_{\alpha \in \varphi^+}$ with the base chamber containing the base chamber of $\mathcal{H}$. Then we have

$$P(\mathcal{H}, K) = \sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon(\varphi) \cdot P(\mathcal{H}_\varphi, K).$$

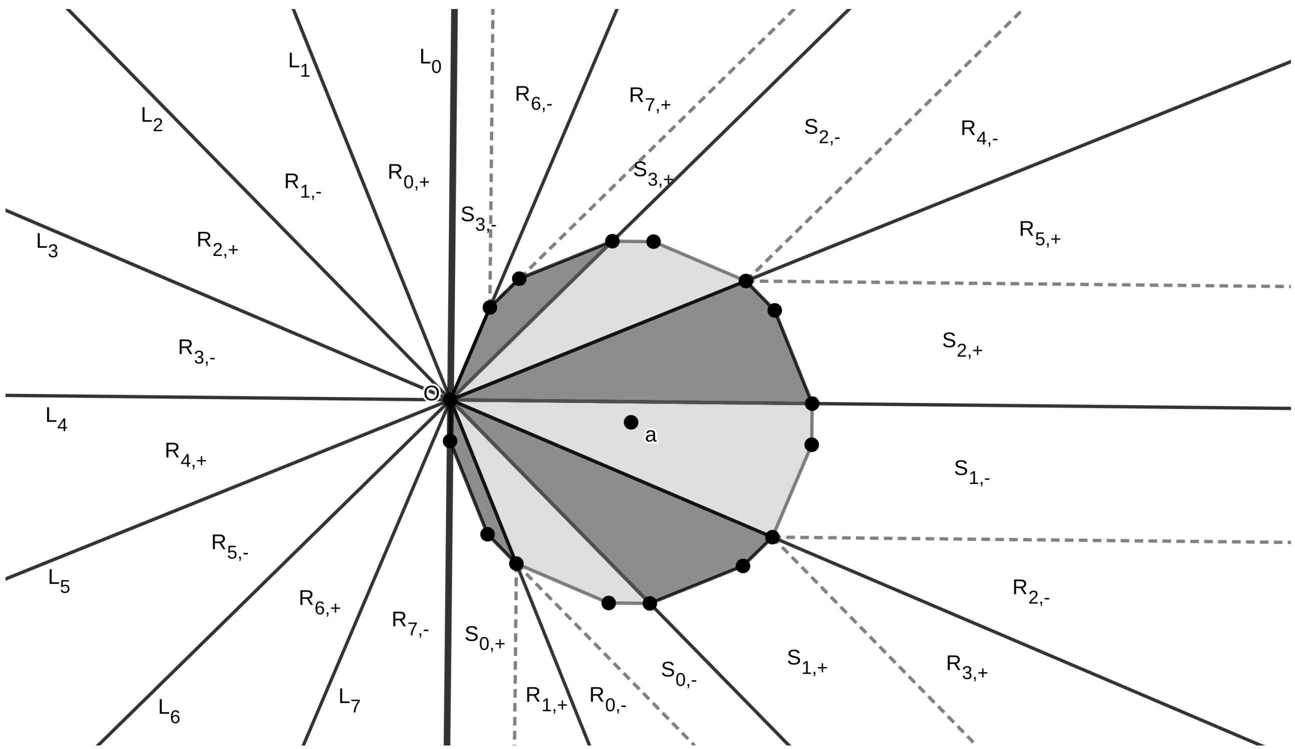
Extremely brief sketch of the proof of the Abstract Pizza Theorem:

Case 1: The 2-structure φ contains a factor of B_2 or $I_2(2^k)$.

Then we prove

$$P(\mathcal{H}_\varphi, K) = 0$$

by reducing it to 2-dimensions and (carefully) moving pieces around.



L0
L1
L2
L3
L4
L5
L6
L7
R6,-
R7,+
S2,-
R4,-
R1,-
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S3,-
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a
R4,+
S1,-
R5,-
R2,-
R6,+
R7,-
S0,+
S1,+
R3,+
S0,-
R1,+
R0,-

History continued

[1807, Wallace], [1833, Bolyai], [1835, Gerwien]

Two polygons are scissors-congruent if and only if they have the same area.

Case 2: The 2-structure φ has type A_1^n.

$$P(\mathcal{H}_\varphi, K + a) = \left[\prod_{i=1}^{n} (0, 2(a, e_i)e_i]\right],$$

where $\varphi^+ = \{e_1, \ldots, e_n\}$.

Thus $P(\mathcal{H}, K + a)$ is a signed sum of parallelotopes.

This sum is zero by an extension of the Wallace–Bolyai–Gerwien theorem to parallelotopes. $\square$

Let V_i denote the ith mixed volume.

Corollary. With the same assumptions as in the abstract pizza theorem:

$$\sum_{T \in \mathscr{T}} (-1)^T V_i((K+a) \cap T) = 0.$$

Other pizza results and open problems.

Returning to classical pizza quantity, that is, volume.

Also returning to balls $\mathbb{B}(a,R) = \{x \in V : \|x-a\| \leq R\}$.

Theorem. Let $\mathcal{H} = \{H_e\}_{e\in E}$ be a Coxeter arrangement in an n-dimensional space V. Assume that $|\mathcal{H}| \equiv n \bmod 2$, $|\mathcal{H}| > n$ and $0 \in \mathbb{B}(a,R)$. Then

$$P(\mathcal{H}, \mathbb{B}(a,R)) = 0.$$

Returning to classical pizza quantity, that is, volume.

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Theorem. Let $\mathcal{H} = \{H_e\}_{e \in E}$ be a Coxeter arrangement in an n-dimensional space V. Assume that $|\mathcal{H}| \equiv n \bmod 2$, $|\mathcal{H}| > n$ and $0 \in \mathbb{B}(a, R)$. Then

$$P(\mathcal{H}, \mathbb{B}(a, R)) = 0.$$

SURGEON GENERAL'S WARNING:

This result contains

CALCULUS.

Note: The $-\mathrm{id}_V \in W$ condition implies $|\mathcal{H}| \equiv n \bmod 2$.

This result also holds for types A_n where $n \equiv 0, 1 \bmod 4$ and E_6.

Open problem: Find a dissection proof.

Open problem:

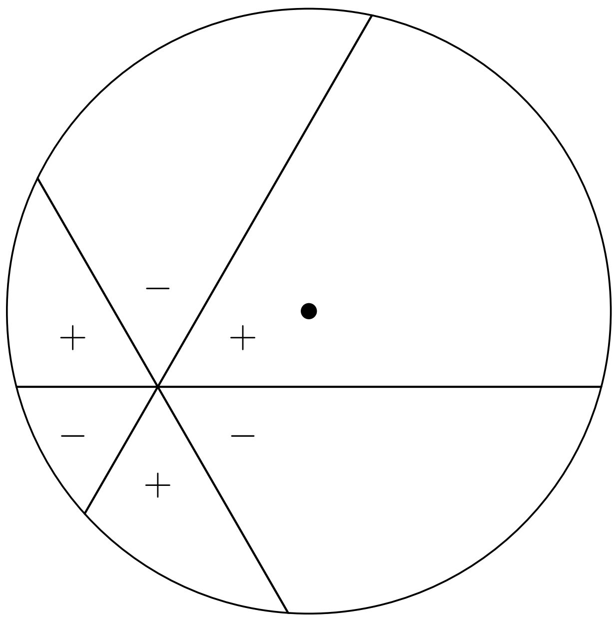
– A_n where $n \geq 3$, $n \equiv 2, 3 \bmod 4$

– D_n where $n \geq 5$, $n \equiv 1 \bmod 2$

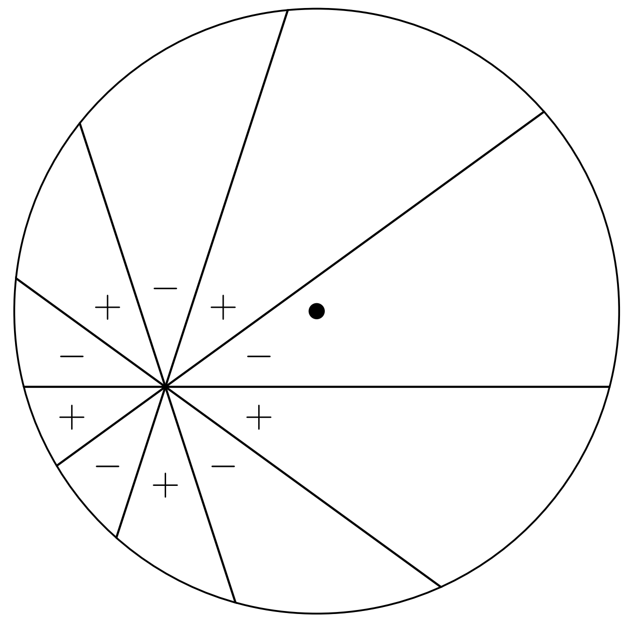
[Mabry and Deiermann]
For $\mathcal{H}$ of type $I_2(m)$, $m \geq 3$, m odd, $0 \in \mathbb{B}(a, R)$ and $a \in T$

$$(-1)^{(m+1)/2} \cdot (-1)^T \cdot P(\mathcal{H}, \mathbb{B}(a, R)) > 0$$

$m \equiv 3 \bmod 4$



$m \equiv 1 \bmod 4$



[Hirschhorn[5]]

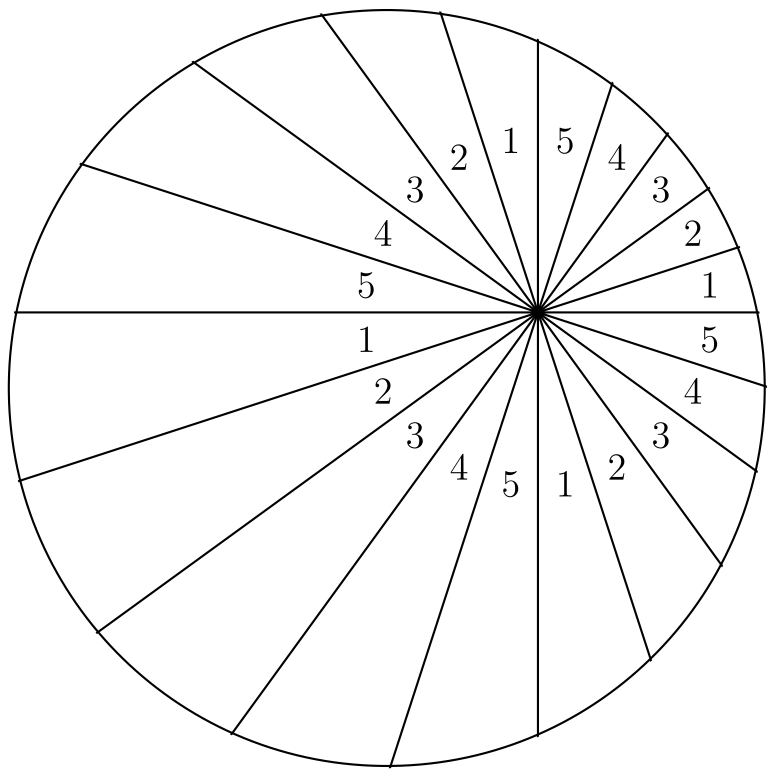
p people sharing a pizza.

Dihedral arrangement of type $I_2(2p)$

Number of slices $4p$

Every person takes every pth slice

Distribution is fair



Open problem:

$p \geq 3$ people in $d \geq 3$ dimensions

Which arrangements guarantee a fair division of $\mathbb{B}(a, R)$?

One solution for $p = d = 4$.

$$\mathcal{H}_1 = \{x_i = \pm x_j : 1 \leq i < j \leq 4\}$$

$$\mathcal{H}_2 = \{x_i = 0 : 1 \leq i \leq 4\} \cup \{x_1 \pm x_2 \pm x_3 \pm x_4 = 0\}$$

Both $\mathcal{H}_1$ and $\mathcal{H}_2$ have type D_4.

The type of $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$ is F_4.

T chamber of $\mathcal{H}$.
Let T_i be the unique chamber in $\mathcal{H}_i$ containing T.

$$(-1)^T = (-1)^{T_1} \cdot (-1)^{T_2}$$

For T a chamber of $\mathcal{H}$ give the slice $T \cap K$ to person $((-1)^{T_1}, (-1)^{T_2})$

Let V_{s_1,s_2} be the amount person (s_1, s_2) receives.

$\mathcal{H}_1$ satisfies pizza theorem $\implies V_{1,1} + V_{1,-1} = 1/2$ pizza

$\mathcal{H}_2$ satisfies pizza theorem $\implies V_{1,1} + V_{-1,1} = 1/2$ pizza

$\mathcal{H}$ satisfies pizza theorem $\implies V_{1,1} + V_{-1,-1} = 1/2$ pizza

$\implies V_{1,1} = V_{1,-1} = V_{-1,1} = V_{-1,-1} = 1/4$ pizza

Thank you!

Bon appétit!

References:

Richard Ehrenborg, Sophie Morel and Margaret Readdy, A generalization of combinatorial identities for stable discrete series constants, *The Journal of Combinatorial Algebra* **6** (2022), 109–183.

Richard Ehrenborg, Sophie Morel and Margaret Readdy, Sharing pizza in n dimensions, *Transactions of the American Mathematical Society* **375** (2022), 5829–5857.

Richard Ehrenborg, Sophie Morel and Margaret Readdy, Pizza and 2-structures, preprint 2021.
`https://arxiv.org/abs/2105.07288`

(Just Google “Pizza Ehrenborg”)