① ⟵ ② ⟶ ③

Graph complexes & moduli spaces in tropical geometry

joint with
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① GC (Kontsevich '92, '93).

$\mathbb{Q}$-v.s.

gens (G, η) — G conn. graph, valence$(v) \geq 3$. $\forall v$.

$\eta \in (\Lambda^{|E(G)|} \mathbb{Z}^{E(G)})^\times$.

total ordering of $E(G)$ up to even permutation

$\deg(G,\eta) = |E(G)| - g(G)$ $\qquad g(G) := b_1(G) = |E(G)| - |V(G)| + 1.$

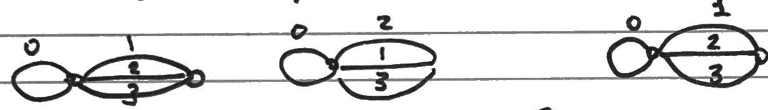
relns : ⓐ $(G, -\eta) = -(G, \eta)$

ⓑ $(G, \eta) = (G', \varphi_* \eta)$ $\quad$ if $\varphi : G \xrightarrow{\cong} G'$

diff $\quad d(G,\eta) = \sum_{i=0}^{p} (-1)^i (\underbrace{G/e_i}_{=0 \text{ if } e_i \text{ loop edge.}}, e_0 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_p)$ $\qquad \deg(d) = -1$

$\| \quad (G, \eta = e_0 \wedge \cdots \wedge e_p)$

$GC = \bigoplus_g GC^{(g)}$

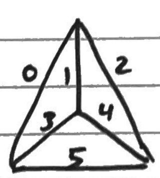
Ex. 

$(G,\eta) \overset{ⓐ}{=} -(G,-\eta) \overset{ⓑ}{=} -(G,\eta) = 0$

since G has non-alternating automorphism i.e. $\mathrm{Aut}(G) \ni \alpha$ which acts on $E(G)$ via an odd perm.

★ count graphs w/ alt automorphisms.

PROBLEM Study $H_\bullet(GC) = \dfrac{\ker d}{\operatorname{im} d}$. (Why?) $\qquad$ Willwacher $H_0(GC) = (\mathfrak{grt}_1)^\vee$

$0 \neq$ [] $\in H_0(GC).$

K_4

UPDATE, Alex Wilson & Hugh Thomas both pointed out that there is a simple combinatorial explanation. the general question of a combinatorial understanding of why $H_k(GC) = 0$ for $k < 0$ is still open.

$0 = [K_k] \in H_{\underbrace{\binom{k}{2} - 2\left(\binom{k}{2} - k + 1\right)}} (GC)$ $\quad k > 4$. $\qquad$ open: find a comb proof.

complete graphs

$(K_k, \eta) \neq 0$ iff k even

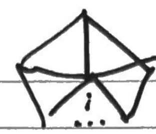
Willwacher-2015 $H_{<0}(GC) = 0$.

CGP, Harer. -2019

$K_4 = W_3$

W_g wheel graph of genus g

W_g alternating iff g odd.



$$0 \neq \underset{g \text{ odd}}{[\, W_g \,]} \in H_0(GC)$$ (!) Rossi-Willwacher, Brown-Schnetz

② Δ-complexes (e.g. Hatcher)

$\forall p \geq 0$ a set X_p "names of p-simplices"

$\forall q \leq p$, $\{0,\dots,q\} \xhookrightarrow{i} \{0,\dots,p\}$ order preserving injection

a ~~map of~~ function $X_p \xrightarrow{i^*} X_q$

i.e. Δ-cx $= \mathrm{OI}^{op} \longrightarrow \mathrm{Set}$

cat. ob. $\{0,\dots,p\}$

morph. order-preserving injections.

$|\cdot| : \Delta\text{-cx} \longrightarrow \mathrm{Top}$.

geometric realization.

Ex. $\overset{v}{\bullet}\!\xrightarrow{e}\!\overset{w}{\circ}$ $\quad I$

$I_1 = \{e\}$

$I_0 = \{v, w\}$.

Symmetric Δ-complex (CGP)

$\forall p \geq 0$ a set X_p ....

$\forall q \leq p$, $\{0,\dots,q\} \xhookrightarrow{j} \{0,\dots,p\}$

a set map $X_p \xrightarrow{j^*} X_q$.

symmetric Δ-cx is: a functor $\mathrm{FI}^{op} \longrightarrow \mathrm{Set}$

ob. $\{0,\dots,p\}$.

morphisms: any injections

$|\cdot| : \Delta\text{-cx} \longrightarrow \mathrm{Top}$.

H $\overset{v}{\bullet}\!\overset{e}{—}$

$H_1 = \{e\}$

$H_0 = \{v\}$.

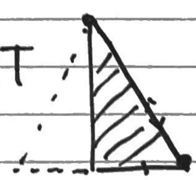
$\{0,1\} \xrightarrow{\sigma} \{0,1\}$

$\sigma^* : H_1 \to H_1$

$e \mapsto e$

C. Yun DMT for sym Δ-complexes.

HT



2 vertices

3 edges

3 faces.

PROP. (CGP) define, for X symmetric Δ-cx,

$S_{p+1} \curvearrowright X_p$.

$$C_p(X;\mathbb{Q}) := \mathbb{Q}_{sgn} \otimes_{S_{p+1}} \mathbb{Q}X_p = \frac{\mathbb{Q}X_p}{x = (\mathrm{sgn}\,\sigma)\sigma x} \quad \sigma \in S_{p+1},\ x \in X_p$$

then

$$\cdots \to C_{p+1}(X;\mathbb{Q}) \to C_p(X;\mathbb{Q}) \xrightarrow{\partial} C_{p-1}(X;\mathbb{Q}) \to$$

$$\simeq$$

$$C^{sing}_{p+1}(X;\mathbb{Q}) \to C^{sing}_p(X;\mathbb{Q}) \longrightarrow C^{sing}_{p-1}(X;\mathbb{Q}) \to$$

$x \in X_p$ $\quad x \neq 0$

iff $\mathrm{Aut}(x) \leq S_{p+1}$ ~~is~~ is alternating.

$g \geq 2$

③ Δ_g moduli space of tropical curves of genus g

parameter space.

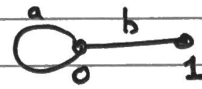
↳ (G, w, ℓ) G conn graph

$w: V(G) \to \mathbb{Z}_{\geq 0}$

$\ell: E(G) \to \mathbb{R}_{>0}$ $\quad \sum \ell(e) = 1$.

s.t. the graph G^w has vertices val ≥ 3.

$\leadsto$ place $w(v)$ loops at each $v \in V$.

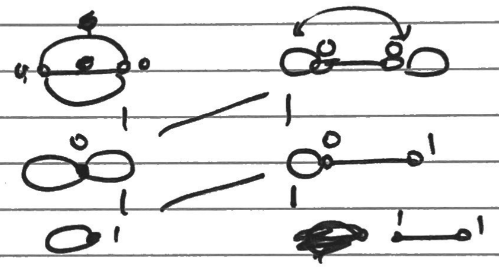


$a + b = 1$.

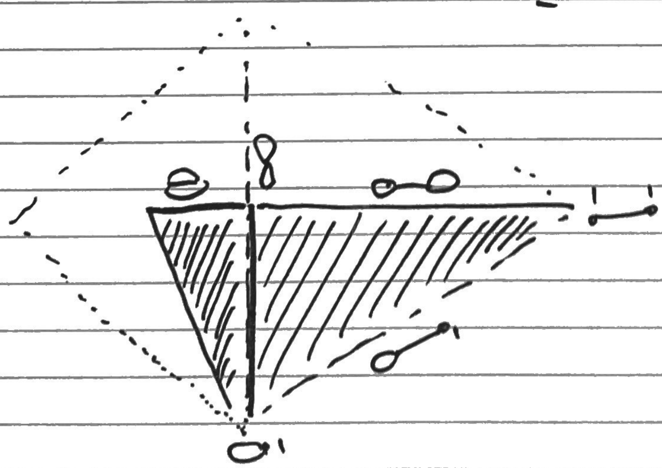
genus $g(G, w, \ell) = b_1(G^w)$.

Fix g. $\exists$ (G, w) finite

e.g. $g = 2$



$\Delta_g = \{\text{tropical curves of genus } g\}/\cong$



PROP. Δ_g is a symmetric Δ-cx.

THM. (CGP) $\quad H_\cdot(GC) \underset{Ⓐ+Ⓑ}{\cong} \bigoplus_g \tilde{H}_\cdot(\Delta_g; \mathbb{Q}) \xleftarrow{\text{Deligne}} \bigoplus_g H^\cdot(M_g; \mathbb{Q})$

Ⓐ $\Delta_g^{\text{weight}} = \{\text{trop curves with pos. weights}\}$ is contractible

Ⓑ cellular hom. thy symmetric Δ-complexes.