Equivariant log-concavity of independence sequences of clawfree graphs

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(1) the Hodge decomposition on $H^{\bullet}(X;\mathbb{C})$

$$H^i(X;\mathbb{C}) = \bigoplus_{p+q=i} \Sigma H^{p,q}(X;\mathbb{C})$$

where $H^{p,q}(X;\mathbb{C})$ = classes of closed forms of type (p,q).

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surjective for i no smaller than n.

Example $(\mathbb{P}^1)^3$

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is injective for i no greater than $n-1$

surjective for i no smaller than n by HLT.

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	$x_1 x_2 x_3$		6
$x_1 x_2$	$x_2 x_3$	$x_1 x_3$	4
x_1	x_2	x_3	2
	1		0

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$$\begin{tikzcd}
 & x_1 x_2 x_3 & & & 6 \\
x_1 x_2 & x_2 x_3 & x_1 x_3 & & 4 \\
x_1 & x_2 & x_3 & & 2 \\
 & 1 \arrow[ul] \arrow[u] \arrow[ur] & & & 0
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x_1 \arrow[u] \arrow[urr] & x_2 & x_3 & 2 \\
 & 1 \arrow[ul] \arrow[u] \arrow[ur] & & 0
\end{tikzcd}$$

$$\begin{aligned}
& x_1(x_1 + x_2 + x_3) \\
&= x_1^2 + x_1 x_2 + x_1 x_3 \\
&= 0 + x_1 x_2 + x_1 x_3
\end{aligned}$$

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HLT

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Application to combinatorics

G is claw-free

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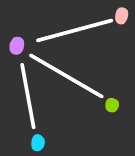

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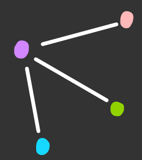
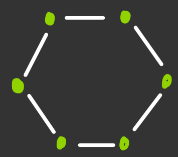


Example cycles



K_n

the line graph of a graph

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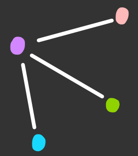

A vertex set $S \subseteq V(G)$ is independent

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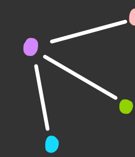

A vertex set $S \subseteq V(G)$ is independent

if vertices in S are pairwise nonadjacent.

A generalized Petersen graph.

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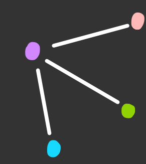

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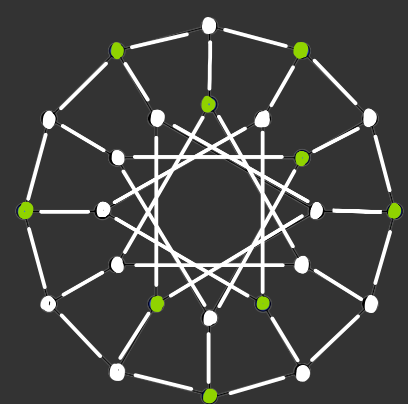

A generalized Petersen graph

$$I_k = \{\text{Independent sets of size } k \text{ on } G\}$$

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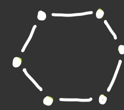
Example



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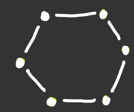
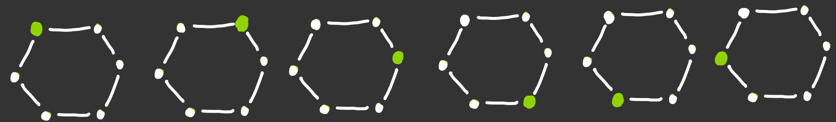
Example

I_0



I_1



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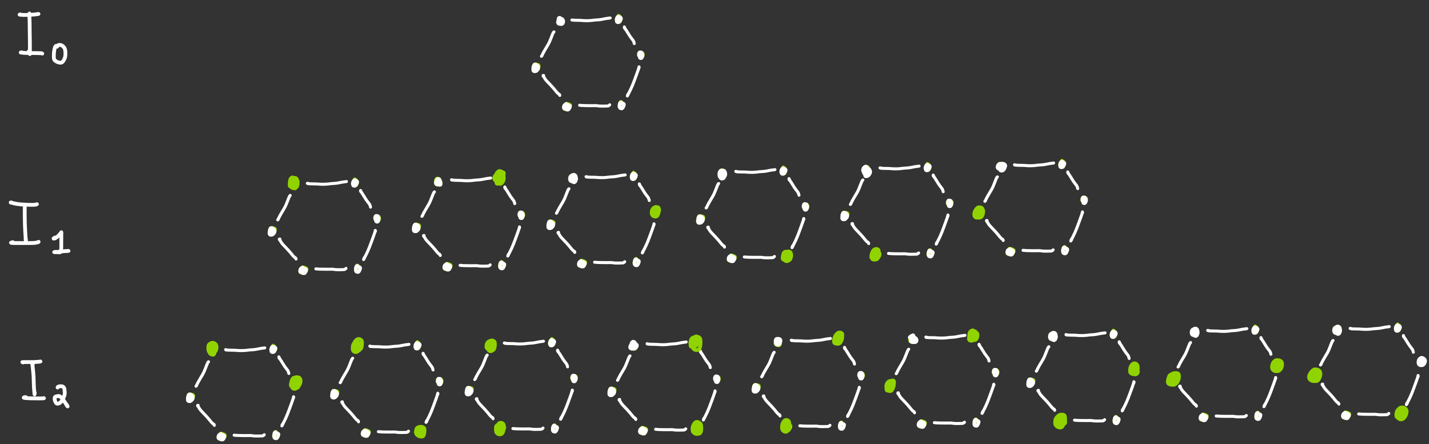

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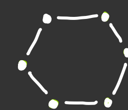

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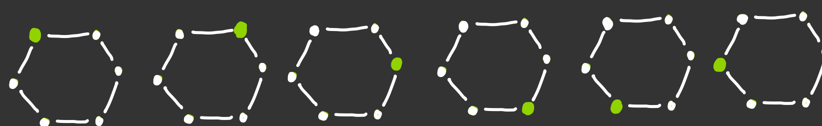
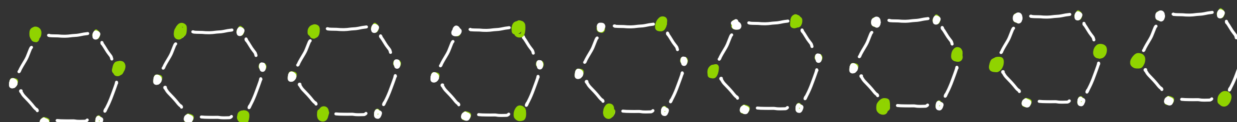
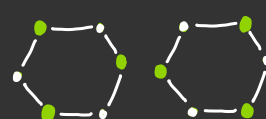


I_2



I_3



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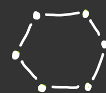

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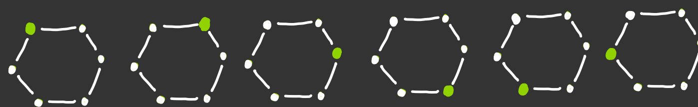
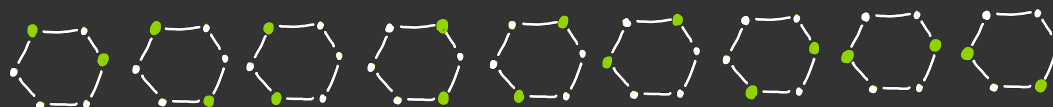


I_2



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$$V_\bullet = \mathrm{Span}_{\mathbb{C}}\{\text{Independent set } I \text{ on } G\} = \bigoplus_{k=|I|} V_k$$

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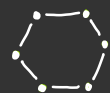

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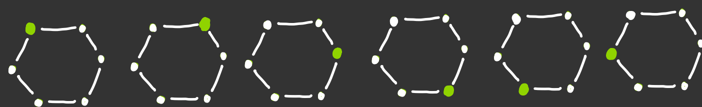
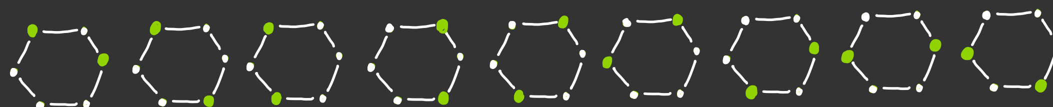


I_2



I_3

Aut(G)

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Example

I_0

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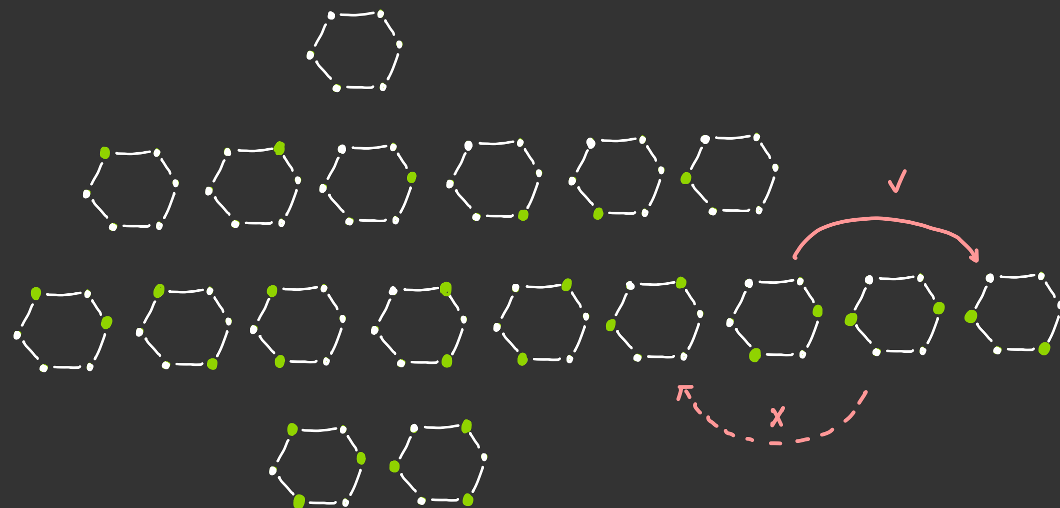
I_2

I_3



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Question

For any $1 \leq k \leq \ell$, is it true that

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as an $\mathrm{Aut}(G)$-representation?

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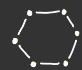
Hamidoune '90

Chudnovsky-Seymour '07, Newton 1707

THM (L. '22) If G is clawfree, then the graded $\mathrm{Aut}(G)$-representation $V_\bullet(G)$ is equivariantly log-concave.

Example $\quad V_\bullet = \mathrm{Span}_{\mathbb{C}}\{\text{Independent set } I \text{ on } G\} = \bigoplus_{k=|I|} V_k$

$V_0 = \mathrm{Span}_{\mathbb{C}}\{$  $\}$

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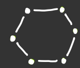
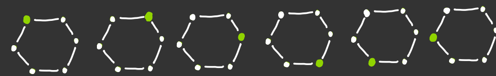
$V_0 = \mathrm{Span}_{\mathbb{C}} \{$  $\}$

$V_1 = \mathrm{Span}_{\mathbb{C}} \{$  $\}$

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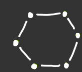
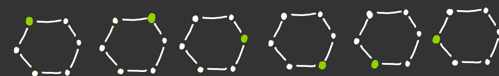
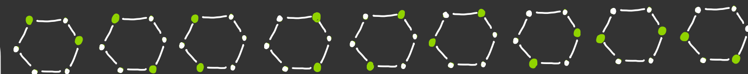
$V_0 = \mathrm{Span}_{\mathbb{C}}\{$  $\}$

$V_1 = \mathrm{Span}_{\mathbb{C}}\{$  $\}$

$V_2 = \mathrm{Span}_{\mathbb{C}}\{$  $\}$

$V_3 = \mathrm{Span}_{\mathbb{C}}\{$  $\}$

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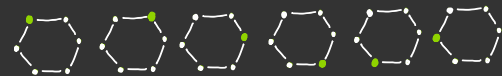
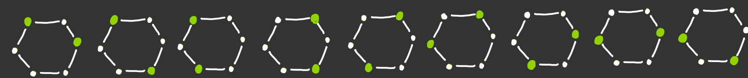
$V_1 = \mathrm{Span}_{\mathbb{C}} \{$  $\}$

$V_2 = \mathrm{Span}_{\mathbb{C}} \{$  $\}$

$V_3 = \mathrm{Span}_{\mathbb{C}} \{$  $\}$

Is it true that

$$V_0 \otimes V_2 \xhookrightarrow{\mathrm{Aut}(G)} V_1 \otimes V_1$$

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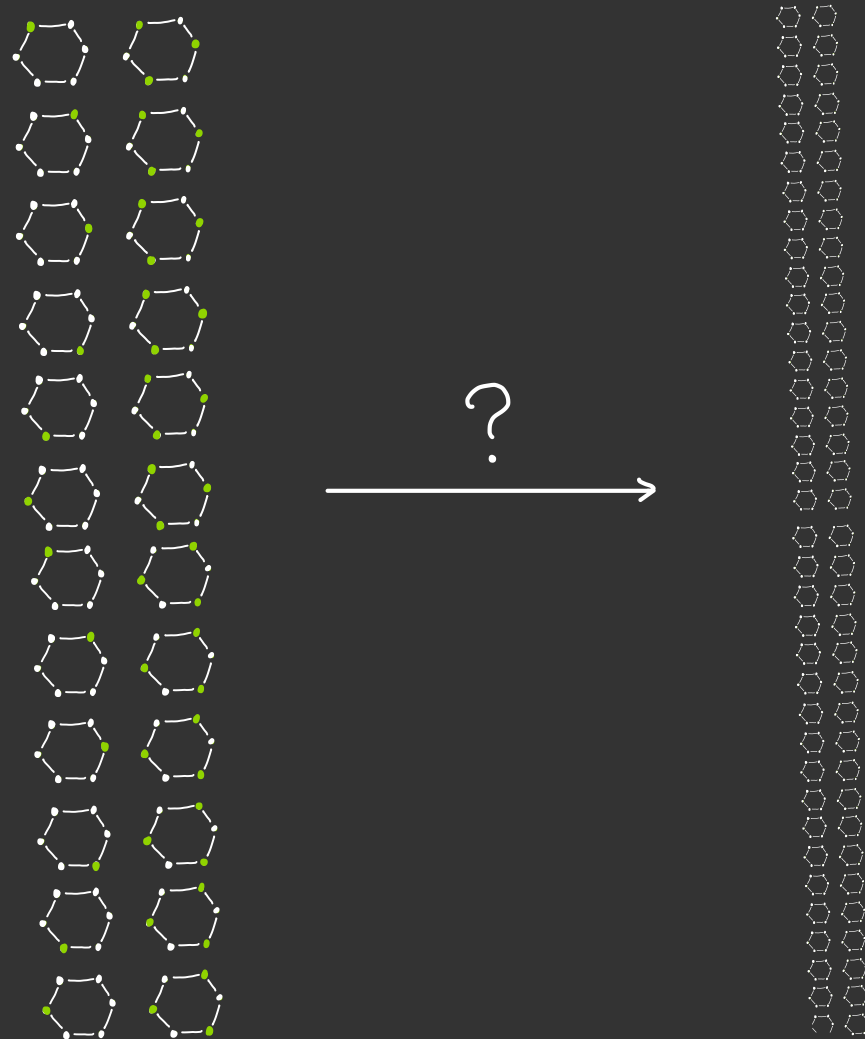
$$V_1 \otimes V_3 \xhookrightarrow{\mathrm{Aut}(G)} V_2 \otimes V_2 \qquad ?$$



Strategy For each $1 \leq k \leq \ell$, decompose

$$V_{k-1} \otimes V_{\ell+1} = \bigoplus_{\Gamma} V_{\Gamma}$$

$$V_k \otimes V_\ell = \bigoplus_{\Gamma'} V_{\Gamma'}$$

where each Γ is a certain induced subgraph of G.

Strategy For each $1 \leq k \leq \ell$,

decompose $V_{k-1} \otimes V_{\ell+1} = \bigoplus_{\Gamma} V_{\Gamma}$

$$V_k \otimes V_\ell = \bigoplus_{\Gamma'} V_{\Gamma'}$$

such that for each Γ,

$$V_\Gamma \xrightarrow{\sim} H^i((\mathbb{P}^1)^n)$$

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$$V_{k-1} \otimes V_{\ell+1} = \bigoplus_{\Gamma} V_{\Gamma}$$
$$V_k \otimes V_\ell = \bigoplus_{\Gamma'} V_{\Gamma'}$$

such that for each Γ,

① $$V_\Gamma \xrightarrow{\sim} H^i((\mathbb{P}^1)^n)$$

for some $i \leq n$. Both i and n depend on a certain induced subgraph Γ.

Strategy For each $1 \leq k \leq \ell$,

decompose
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such that for each Γ,

① $V_\Gamma \xrightarrow{\sim} H^i((\mathbb{P}^1)^n)$

③ $\exists$ Aut(G)-equiv map $\Phi_{k,\ell}$ $\quad \Phi_\Gamma \downarrow$

② $V_{\Gamma'} \xrightarrow{\sim} H^{i+2}((\mathbb{P}^1)^n)$

for some $i \leq n$. Both i and n depend on Γ.

Strategy For each $1 \leqslant k \leqslant \ell$,

decompose $V_{k-1} \otimes V_{\ell+1} = \bigoplus_{\Gamma} V_{\Gamma}$

$$V_k \otimes V_\ell = \bigoplus_{\Gamma'} V_{\Gamma'}$$

such that for each Γ,

(1)

(3) $\exists$ Aut(G)-equiv map $\Phi_{k,\ell}$

(2)

$$\begin{tikzcd}
V_{\Gamma} \arrow[r, "\sim"] \arrow[d, "\Phi_{\Gamma}"'] \arrow[dr, phantom, "\circlearrowright"] & H^{i}((\mathbb{P}^1)^n) \arrow[d, "w \cup -"] \\
V_{\Gamma'} \arrow[r, "\sim"] & H^{i+2}((\mathbb{P}^1)^n)
\end{tikzcd}$$

for some $i \leqslant n$. Both i and n depend on Γ.

Strategy For each $1 \leq k \leq \ell$,

decompose $V_{k-1} \otimes V_{\ell+1} = \bigoplus_{\Gamma} V_{\Gamma}$

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such that for each Γ,

① ③ $\exists$ Aut(G)-equiv map $\Phi_{k,\ell}$ ②

$$\begin{tikzcd}
V_{\Gamma} \arrow[r, "\sim"] \arrow[d, hook, "\Phi_{\Gamma}"'] \arrow[dr, phantom, "\circlearrowright"] & H^{i}((\mathbb{P}^1)^n) \arrow[d, hook, "\text{wu-}"] \\
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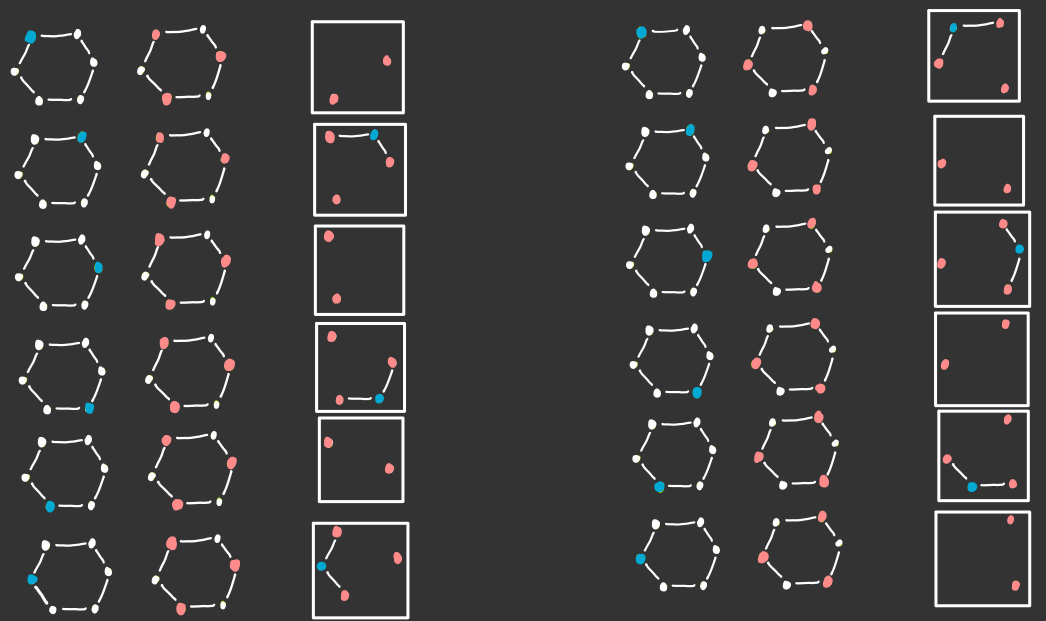
\end{tikzcd}$$

for some $i \leq n$. Both i and n depend on Γ.

The hard Lefschetz theorem implies the injectivity of Φ_{Γ}.

1. Take the symmetric difference of each pair, and induce a subgraph Γ.



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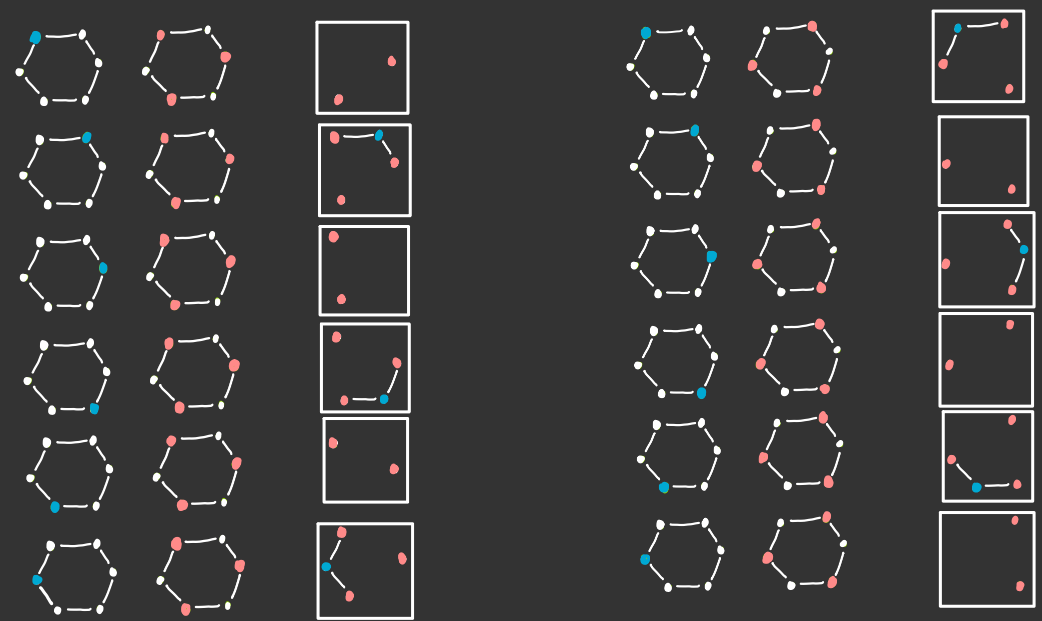
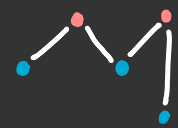


Property of Γ Each component is either a path or a cycle of even size.

2. Consider only paths of odd # of vertices

"even paths"



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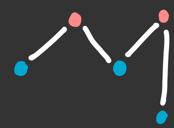

blue path

pink path

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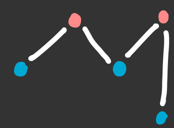


blue path

pink path

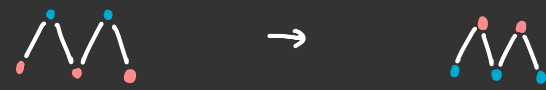
B = # of blue paths

P = # of pink paths

3. Swap colors in one path

pink path

blue path



4. Swap colors in every even path in Γ.

Form a new pair of independent sets of size k, ℓ.

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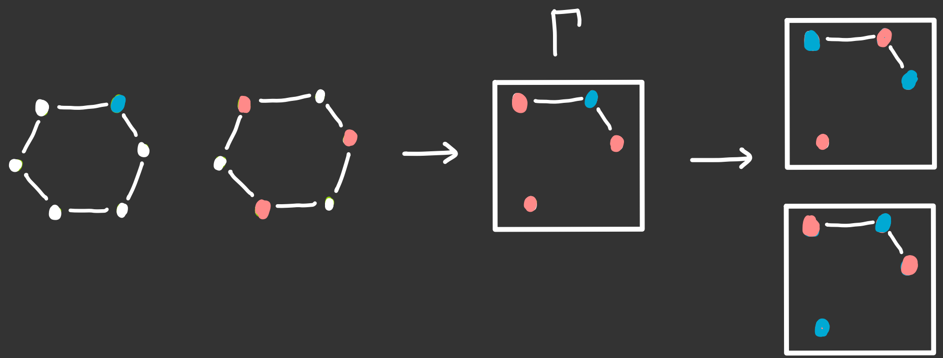

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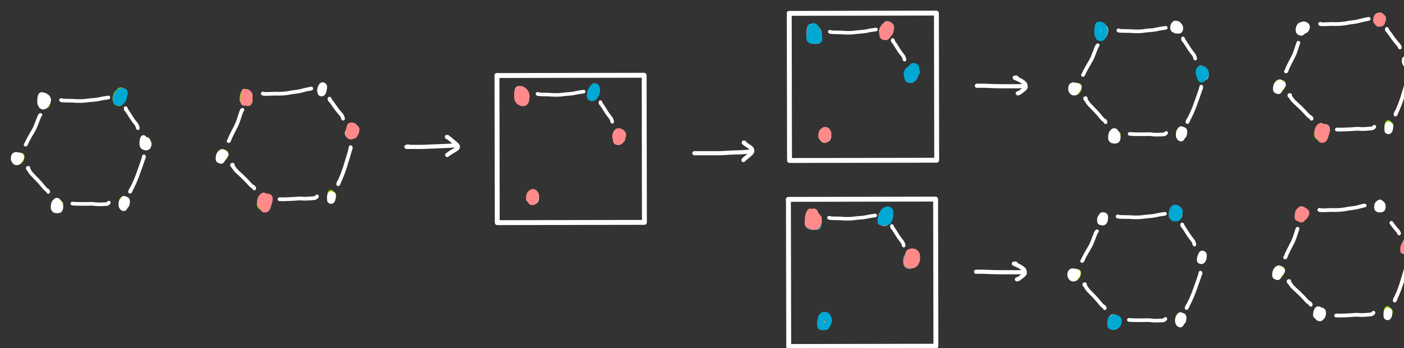

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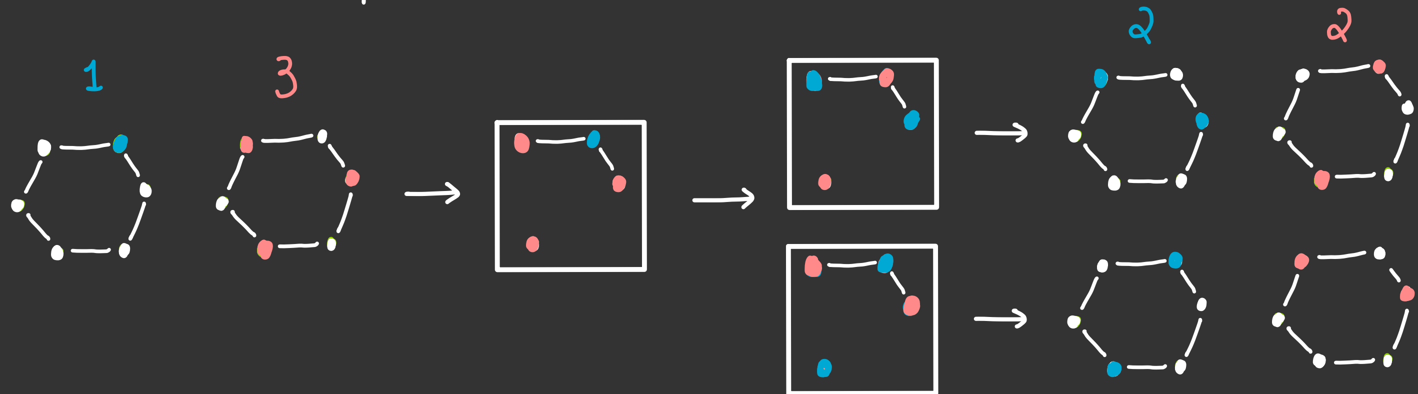

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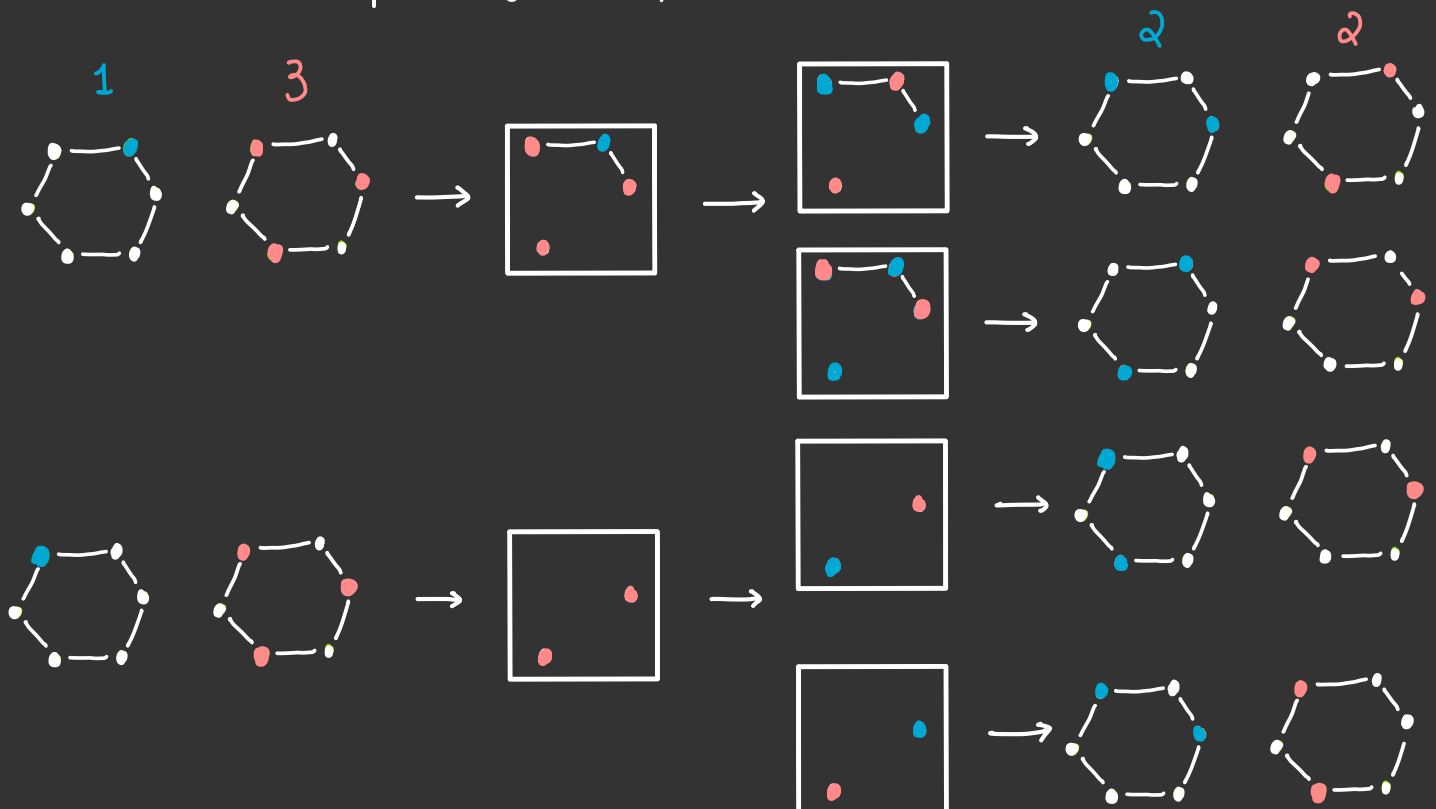

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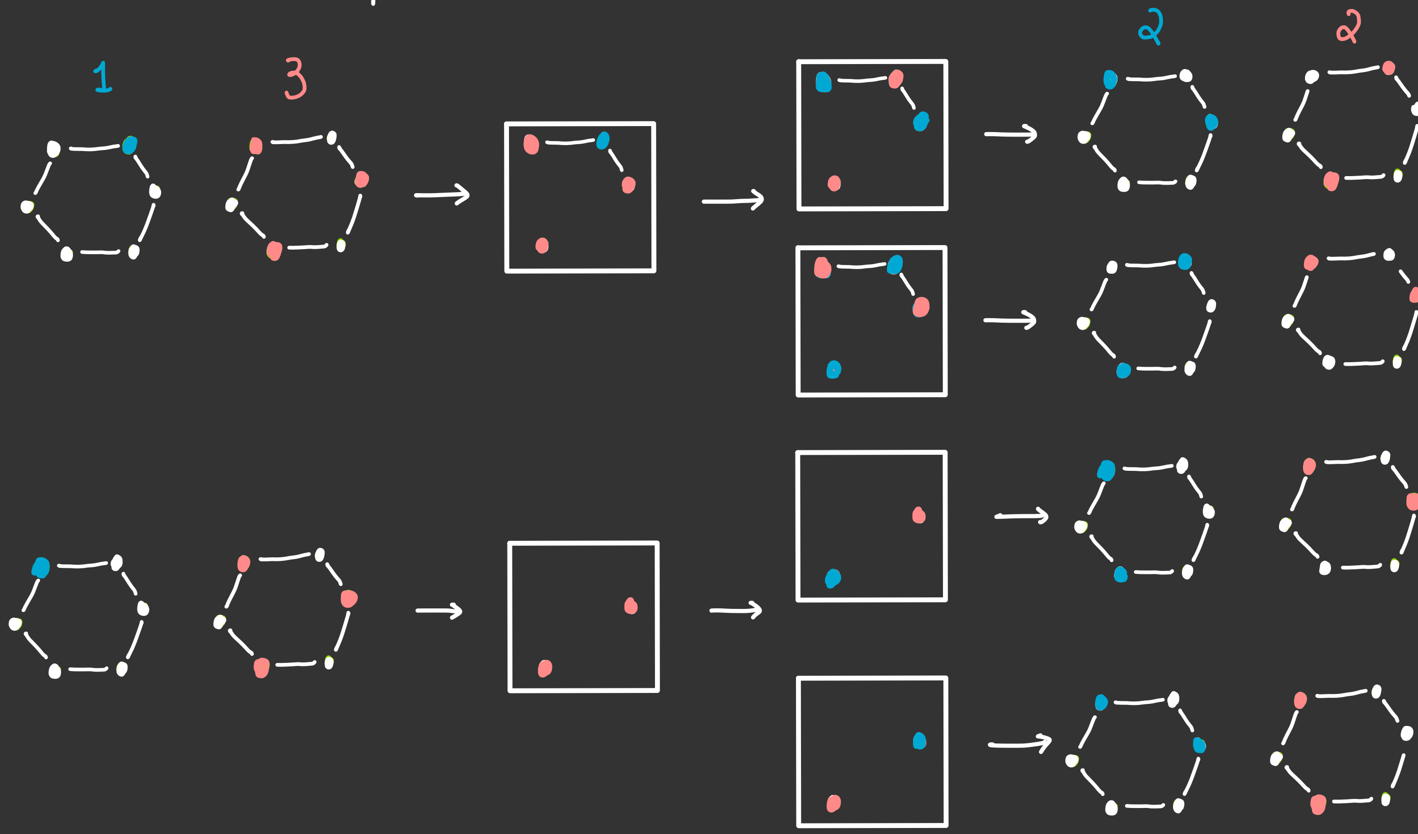
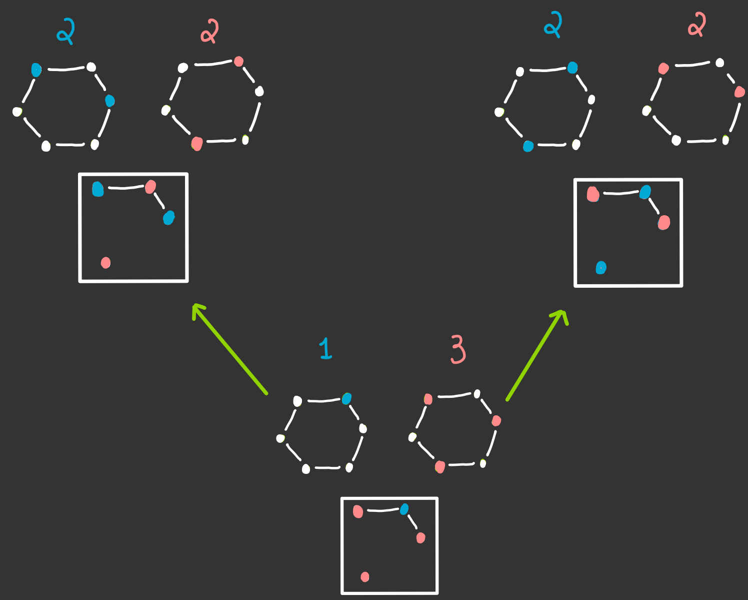


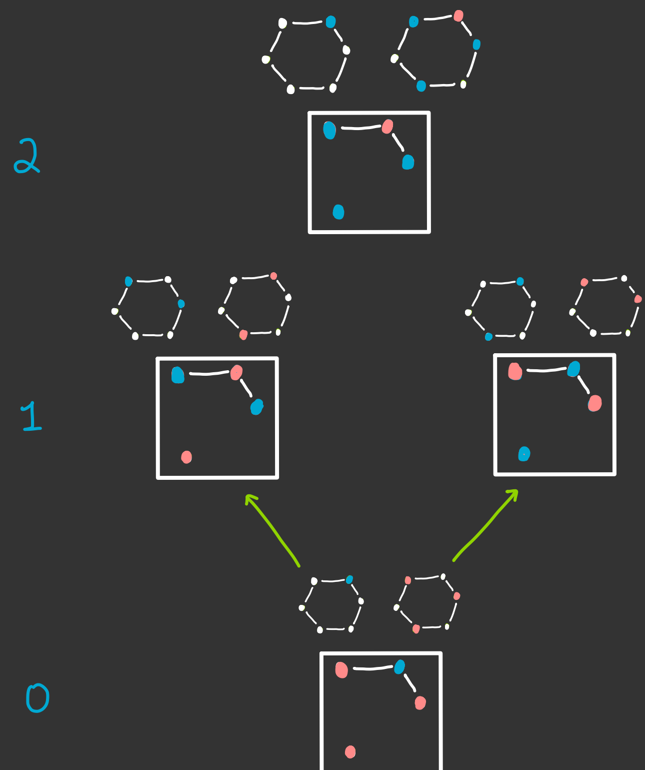
5. Define $\Phi_{k,\ell} : V_{k-1} \otimes V_{\ell+1} \longrightarrow V_k \otimes V_\ell$

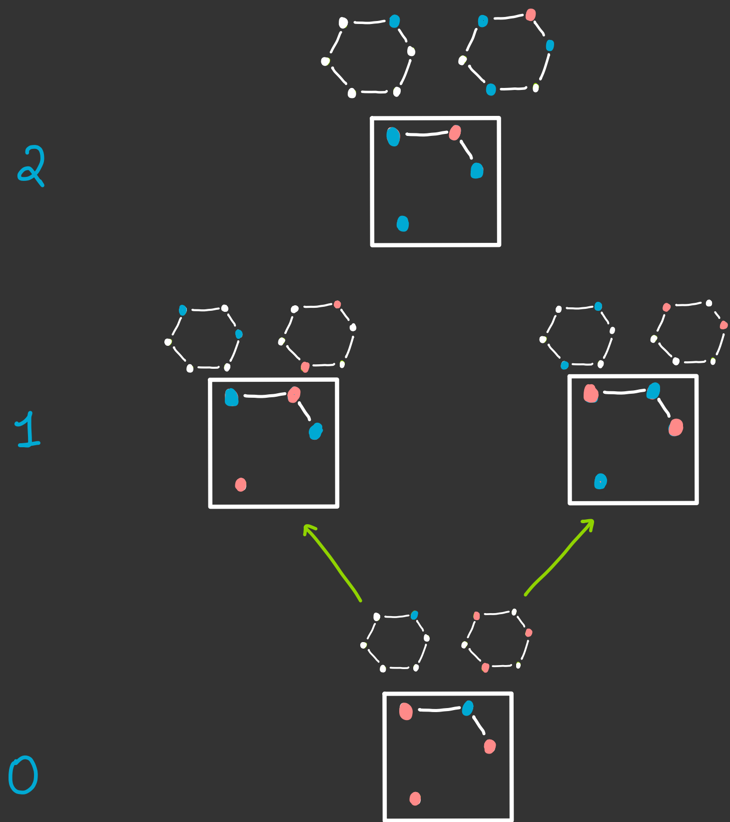
$$I \otimes J \longmapsto \sum' I' \otimes J'$$

where $I' \otimes J'$ are pairs obtained as in 1-4.



5. Consider the Boolean algebra formed on all even paths in Γ, graded by B.





$\simeq$

$H^\bullet((\mathbb{P}^1)^2)$

$$\begin{tikzcd}
 & x_1 x_2 & & 4 \\
x_1 & & x_2 & 2 \\
 & 1 \arrow[ul] \arrow[ur] & & 0
\end{tikzcd}$$

$$\Phi_{k,\ell}: V_{k-1} \otimes V_{\ell+1} \longrightarrow V_k \otimes V_\ell$$

$$I \otimes J \longmapsto \sum I' \otimes J'$$

where $I' \otimes J'$ are pairs obtained as in 1–4 restricted to each induced subgraph Γ.

$=$

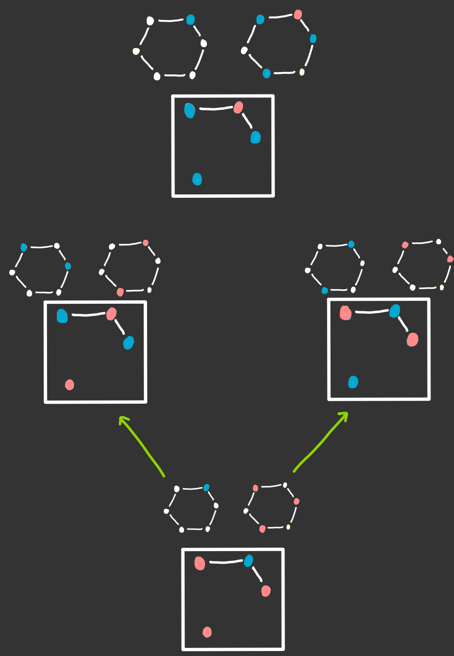
The hard Lefschetz operator on $(\mathbb{P}^1)^n$

$$w: H^i \longrightarrow H^{i+2}$$

multiplication by

$x_1 + x_2 + \cdots + x_n$.

where $i = B$ in Γ, $n = B + P$ in Γ.



$\simeq$

$$H^\bullet\left((\mathbb{P}^1)^2\right)$$

$$\begin{tikzcd} & x_1 x_2 & \\ x_1 & & x_2 \\ & 1 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

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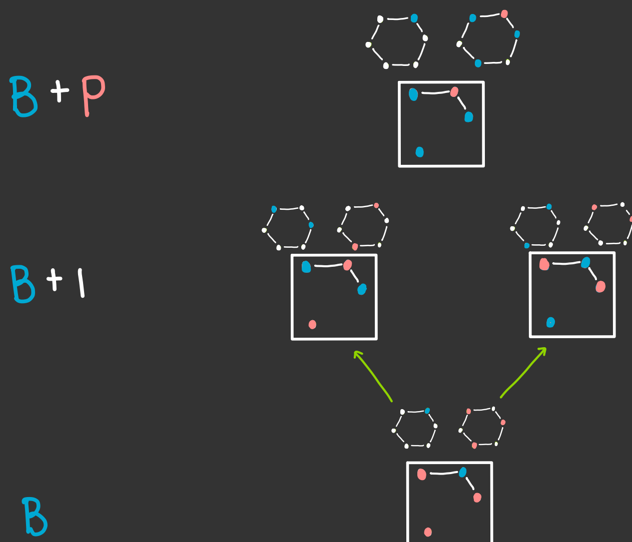


$\simeq$

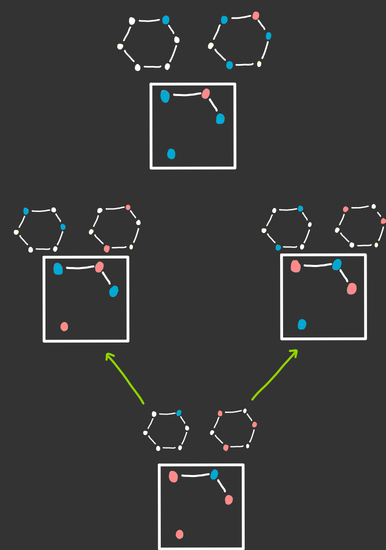
$$H^\bullet((\mathbb{P}^1)^2)$$

$$\begin{tikzcd}[column sep=small] & x_1 x_2 & \\ x_1 & & x_2 \\ & 1 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

B + P

B + 1

B



$$H^\bullet((\mathbb{P}^1)^2) \simeq \begin{tikzcd}[column sep=small] & x_1 x_2 & \\ x_1 & & x_2 \\ & 1 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

We win if B is no greater than $\frac{B+P}{2}$

Property

$$P - B = (\ell + 1) - (k - 1) = (\ell - k) + 2 \geq 2.$$

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$$B = \frac{B + B}{2}$$

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$$B = \frac{B + B}{2} \leq \frac{B + P - 2}{2}$$

Property

$$P - B = (\ell + 1) - (k - 1) = (\ell - k) + 2 \geq 2.$$

$$B = \frac{B + B}{2} \leq \frac{B + P - 2}{2} = \frac{B + P}{2} - 1$$

The Hard Lefschetz Theorem

$$\Longrightarrow \qquad \Phi_{k,\ell} : V_{k-1} \otimes V_{\ell+1} \hookrightarrow V_k \otimes V_\ell$$

$$I, J \longmapsto \sum I', J'$$

where I', J' are independent sets formed by the above steps

THM (L. '22) If G is clawfree, then the graded $\mathrm{Aut}(G)$-representation $V_\bullet(G)$ is equivariantly log-concave.

Remarks

1. ELC is stronger than LC

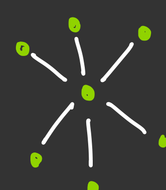
   $\exists\ V_\bullet$ not ELC but $\dim V_\bullet$ is LC.

   L. Kannen : Sterling #'s

2. A related problem

   Erdös' conjecture : the independence sequence of any tree is LC.

   Counter-example for the equivariant statement :



Melody Chan, Chris Eur, Dane Miyata, Nick Proudfoot, Eric Ramos, Lorenzo Vecchi

3 Other known results

Mathern-Miyata-Proudfoot-Ramos '21

$H(\mathrm{Conf}(n,\mathbb{R}^3),\mathbb{Q})$, $H(\mathrm{Conf}(n,\mathbb{C}),\mathbb{Q})$, etc. is S_n-ELC for degrees $m \leq 14$.

Proudfoot-Xu-Young '16

q-binomial coefficients.

L. '22

Matchings for all graphs.

$V_\bullet^n$ given by k-subsets of $[n]$ as S_n representation.