Stembridge codes and Chow rings

Hsin-Chieh Liao

University of Miami

FPSAC'23 at UC Davis

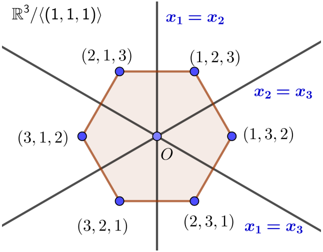
July 19 2023

Eulerian story : Permutohedral variety

The *Permutohedral variety* X_{Σ_n} is the toric variety associated to the normal fan Σ_n of the *permutohedron* Π_n. Its Poincaré polynomial

$$\sum_{j=0}^{n} \dim H^{2j}(X_{\Sigma_n}) t^j = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathsf{exc}(\sigma)}$$

is the *Eulerian polynomial* $A_n(t)$.



$\mathsf{H}^*(X_{\Sigma_n})$ carries a representation of $\mathfrak{S}_n$ induced from $\mathfrak{S}_n$ acting on Σ_n .

Stanley (1989) calculated its Frobenius characteristic and showed that the rep. is a *permutation representation*.

Stembridge Codes (1992)

A *code* is a sequence α over $\{0, 1, \ldots, m\}$ with marks s.t. for each $j = 1, 2, \ldots, m$

- j occurs at least twice in α;
- a mark is assigned to the ith occurrence of j for a unique $i \geq 2$.

Example:

$$11320\hat{2}\hat{3}\hat{1}2$$

We denote a code as (α, f) where $f(j) =$ number of j's in front of the marked j.

Example:

$$11320\hat{2}\hat{3}\hat{1}2 = (\alpha, f)$$

where $\alpha = 113202312$ and $f(1) = 2, f(2) = 1, f(3) = 1$.

The *index* of (α, f): $\operatorname{ind}(\alpha, f) := \sum_{j=1}^{m} f(j)$.

Example:

$$\operatorname{ind}(11320\hat{2}\hat{3}\hat{1}2) = f(1) + f(2) + f(3) = 4.$$

All codes of length 3

m	0	1	1	1	1	1
(α, f)	000	$01\hat{1}$	$10\hat{1}$	$1\hat{1}0$	$1\hat{1}1$	$11\hat{1}$
$\operatorname{ind}(\alpha, f)$	0	1	1	1	1	2

Stembridge Codes (1992)

A *code* is a sequence α over $\{0, 1, \ldots, m\}$ with marks s.t. for each $j = 1, 2, \ldots, m$

- j occurs at least twice in α;
- a mark is assigned to the ith occurrence of j for a unique $i \geq 2$.

Example:

$$11320\hat{2}\hat{3}\hat{1}2$$

We denote a code as (α, f) where $f(j) =$ number of j's in front of the marked j.

Example:

$$11320\hat{2}\hat{3}\hat{1}2 = (\alpha, f)$$

where $\alpha = 113202312$ and $f(1) = 2, f(2) = 1, f(3) = 1$.

The *index* of (α, f): $\operatorname{ind}(\alpha, f) := \sum_{j=1}^{m} f(j)$.

Example:

$$\operatorname{ind}(11320\hat{2}\hat{3}\hat{1}2) = f(1) + f(2) + f(3) = 4.$$

All codes of length 3

m	0	1	1	1	1	1
(α, f)	000	$01\hat{1}$	$10\hat{1}$	$1\hat{1}0$	$1\hat{1}1$	$11\hat{1}$
$\operatorname{ind}(\alpha, f)$	0	1	1	1	1	2

Stembridge Codes (1992)

A *code* is a sequence α over $\{0, 1, \ldots, m\}$ with marks s.t. for each $j = 1, 2, \ldots, m$

- j occurs at least twice in α;
- a mark is assigned to the ith occurrence of j for a unique $i \geq 2$.

Example:

$$11320\hat{2}\hat{3}\hat{1}2$$

We denote a code as (α, f) where $f(j) =$ number of j's in front of the marked j.

Example:

$$11320\hat{2}\hat{3}\hat{1}2 = (\alpha, f)$$

where $\alpha = 113202312$ and $f(1) = 2, f(2) = 1, f(3) = 1$.

The *index* of (α, f): $\mathrm{ind}(\alpha, f) := \sum_{j=1}^{m} f(j)$.

Example:

$$\mathrm{ind}(11320\hat{2}\hat{3}\hat{1}2) = f(1) + f(2) + f(3) = 4.$$

All codes of length 3

m	0	1	1	1	1	1
(α, f)	000	$01\hat{1}$	$10\hat{1}$	$1\hat{1}0$	$1\hat{1}1$	$11\hat{1}$
$\mathrm{ind}(\alpha, f)$	0	1	1	1	1	2

$\mathfrak{S}_n$-action on codes

Let $\mathcal{C}_{n,j}$ be the set of codes of length n with index j.

The $\mathfrak{S}_n$-action $\sigma \cdot (\alpha_1\alpha_2, \ldots \alpha_n, f) \coloneqq (\alpha_{\sigma(1)}\alpha_{\sigma(2)} \ldots \alpha_{\sigma(n)}, f)$ for $\sigma \in \mathfrak{S}_n$ makes $V_{n,j} = \mathbb{C}\mathcal{C}_{n,j}$ a permutation representation of $\mathfrak{S}_n$.

Stembridge (1992) used Stanley's formula for $\mathrm{ch}(H^*(X_{\Sigma_n}))$ to show that for all $0 \leq j \leq n-1$,

$$V_{n,j} \cong_{\mathfrak{S}_n} H^{2j}(X_{\Sigma_n}, \mathbb{C}).$$

He also asked: can we find a permutation basis for $H^*(X_{\Sigma_n})$ that induced the representation?

In our work, we give an explicit permutation basis in terms of Chow rings. And it is clear how $\mathfrak{S}_n$ permutes the basis.

Cohomology ring as Chow ring

Theorem (Danilov 1978, Stembridge 1994)

Fix a lattice $\mathbb{Z}^n$ with standard basis $\{e_i\}_{i=1,\dots,n}$ in a Euclidean n-space E. Let P be a simple n-lattice polytope with normal fan $\Sigma(P)$ in E and P^* be its dual simplicial polytope whose set of vertices is $V(P^*)$. Denote by $K[\partial P^*]$ the Stanley-Reisner ring of ∂P^* over a field K of $\mathrm{char} K = 0$. If a finite group G acts on $\Sigma(P)$ simplicially and freely, then

$$H^*(X_{\Sigma(P)}, K) \cong \frac{K[\partial P^*]}{\langle \theta_1, \dots, \theta_n \rangle} \quad \text{as } G\text{-modules},$$

where

$$\theta_i = \sum_{v \in V(P^*)} \langle v, e_i \rangle x_v \text{ for } i = 1, \dots, n.$$

- The RHS is known as the Chow ring $A(X_{\Sigma(P)})$ of the toric variety $X_{\Sigma(P)}$.

Feichtner-Yuzvinsky basis for $A(X_{\Sigma_n})$

Feichtner, Yuzvinsky (2004) define the Chow ring $D(\mathcal{L}, \mathcal{G})$ of an atomic lattice $\mathcal{L}$ with respect to a certain type of subset $\mathcal{G}$ of $\mathcal{L}$ called a *building set* and find a basis.

When $\mathcal{L} = B_n$ Boolean lattice and $\mathcal{G} = B_n - \{\emptyset\}$, the ring $D(\mathcal{L}, \mathcal{G})$ is known as the Chow ring of the Boolean matroid and is the same as the Chow ring of X_{Σ_n},

$$D(B_n, B_n - \{\emptyset\}) = A(X_{\Sigma_n}).$$

Then by Danilov-Stembridge theorem,

$$D(B_n, B_n - \{\emptyset\}) \cong_{\mathfrak{S}_n} H^*(X_{\Sigma_n}, \mathbb{Q}).$$

In this case, the Feichtner-Yuzvinsky basis is

$$FY(\mathrm{B}_n) := \left\{ x_{F_1}^{a_1} x_{F_2}^{a_2} \dots x_{F_k}^{a_k} : \substack{\emptyset = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \dots \subsetneq F_k \subseteq [n], \\ 1 \le a_i \le |F_i| - |F_{i-1}| - 1} \right\}.$$

e.g. $FY(\mathrm{B}_3) = \{1, x_{12}, x_{13}, x_{23}, x_{123}, x_{123}^2\}$.

The natural $\mathfrak{S}_n$-action on B_n induces an action on $FY(\mathrm{B}_n)$.

Feichtner-Yuzvinsky basis for $A(X_{\Sigma_n})$

Feichtner, Yuzvinsky (2004) define the Chow ring $D(\mathcal{L}, \mathcal{G})$ of an atomic lattice $\mathcal{L}$ with respect to a certain type of subset $\mathcal{G}$ of $\mathcal{L}$ called a *building set* and find a basis.

When $\mathcal{L} = B_n$ Boolean lattice and $\mathcal{G} = B_n - \{\emptyset\}$, the ring $D(\mathcal{L}, \mathcal{G})$ is known as the Chow ring of the Boolean matroid and is the same as the Chow ring of X_{Σ_n},

$$D(B_n, B_n - \{\emptyset\}) = A(X_{\Sigma_n}).$$

Then by Danilov-Stembridge theorem,

$$D(B_n, B_n - \{\emptyset\}) \cong_{\mathfrak{S}_n} H^*(X_{\Sigma_n}, \mathbb{Q}).$$

In this case, the Feichtner-Yuzvinsky basis is

$$FY(\mathsf{B}_n) := \left\{ x_{F_1}^{a_1} x_{F_2}^{a_2} \dots x_{F_k}^{a_k} \, : \, \substack{\emptyset = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \dots \subsetneq F_k \subseteq [n], \\ 1 \leq a_i \leq |F_i| - |F_{i-1}| - 1} \right\}.$$

e.g. $FY(\mathsf{B}_3) = \{1, x_{12}, x_{13}, x_{23}, x_{123}, x_{123}^2\}$.

The natural $\mathfrak{S}_n$-action on B_n induces an action on $FY(\mathsf{B}_n)$.

The $\mathfrak{S}_n$-equivariant bijection

We construct a map $\phi : FY(\mathsf{B}_n) \to \{\text{codes of length } n\}$ that respects the $\mathfrak{S}_n$-action.

Theorem (L.)

The map $\phi : FY(B_n) \to \{\text{codes of length } n\}$ is a bijection that respects the $\mathfrak{S}_n$-actions and takes the degree of the monomials to the index of its image.

Example

The map ϕ sends $x_{13}x_{1235}x_{1234568}^2 \in FY(B_8)$ to a code as follows:

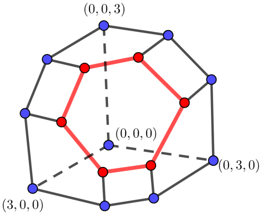
$$1\_\hat{1}\ \_\ \_\ \_\ \_\ \_ \longrightarrow 12\hat{1}\ \_\hat{2}\_\ \_\ \_ \to 12\hat{1}3\hat{2}3\_\hat{3} \to 12\hat{1}3\hat{2}30\hat{3}.$$

The F-Y basis $FY(B_3)$ and Stembridge codes of length 3 :

1	x_{12}	x_{13}	x_{23}	x_{123}	x_{123}^2
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
000	$1\hat{1}0$	$10\hat{1}$	$01\hat{1}$	$1\hat{1}1$	$11\hat{1}$

Binomial Eulerian story: Stellahedral variety

Binomial Eulerian polynomial $\widetilde{A}_n(t) := 1 + t\sum_{k=1}^{n}\binom{n}{k}A_k(t)$



Postnikov, Reiner, Williams (2008) :

$$\widetilde{A}_n(t) = \sum_{j=0}^{n} \dim H^{2j}(X_{\widetilde{\Sigma}_n})t^j$$

where $X_{\widetilde{\Sigma}_n}$ is the toric variety associated to the normal fan $\widetilde{\Sigma}_n$ of the *stellohedron* $\widetilde{\Pi}_n$.

Shareshian, Wachs (2020) :

- Introduced $\widetilde{Q}_n(\mathbf{x},t) := h_n(\mathbf{x}) + t\sum_{k=1}^{n} h_{n-k}(\mathbf{x})Q_k(\mathbf{x},t)$ where $Q_k(\mathbf{x},t)$ is the graded Frobenius series of $H^*(X_{\Sigma_n})$.
- They also showed that $\sum_{j=0}^{n} \mathrm{ch}(H^{2j}(X_{\widetilde{\Sigma}_n}))t^j = \widetilde{Q}_n(\mathbf{x},t)$.

  $\Rightarrow H^*(X_{\widetilde{\Sigma}_n})$ carries a permutation representation of $\mathfrak{S}_n$.

Extended codes

We define an *extended code* to be a code (α, f) s.t.

- α is over $\{0, 1, \ldots, m\} \cup \{\infty\}$ with ∞'s working as 0's in Stembridge codes,
- *index* $\mathrm{ind}(\alpha, f) := \sum_{j=1}^{m} f(j)$ as before, except that $\mathrm{ind}(\infty \ldots \infty) := -1$.

  e.g. $\mathrm{ind}(102\infty\hat{1}32\hat{2}\hat{3}1) = f(1) + f(2) + f(3) = 1 + 2 + 1 = 4.$

Let $\widetilde{\mathcal{C}}_{n,j}$ be the set of extended codes of length n with index j

Example

The extended codes of length 3.
$\widetilde{\mathcal{C}}_{3,-1} = \{\infty\infty\infty\}$, $\widetilde{\mathcal{C}}_{3,0} = \{0\infty\infty, \infty 0\infty, \infty\infty 0, \infty 00, 0\infty 0, 00\infty, 000\}$,
$\widetilde{\mathcal{C}}_{3,1} = \{1\hat{1}\infty, 1\infty\hat{1}, \infty 1\hat{1}, 01\hat{1}, 10\hat{1}, 1\hat{1}0, 1\hat{1}1\}$, $\widetilde{\mathcal{C}}_{3,2} = \{11\hat{1}\}$.

An $\mathfrak{S}_n$-action as before makes $\widetilde{V}_{n,j} = \mathbb{C}\widetilde{\mathcal{C}}_{n,j}$ a permutation representation of $\mathfrak{S}_n$.

Theorem (L. 2022)

For $n \geq 1$, we have $\sum_{j=0}^{n} \mathrm{ch}(\widetilde{V}_{n,j-1}) t^j = \widetilde{Q}_n(\mathbf{x}, t)$.

Is there a permutation basis for $H^*(X_{\widetilde{\Sigma}_n})$ that has similar combinatorial structure as extended codes?

Extended codes

We define an *extended code* to be a code (α, f) s.t.

- α is over $\{0, 1, \ldots, m\} \cup \{\infty\}$ with ∞'s working as 0's in Stembridge codes,
- *index* $\mathrm{ind}(\alpha, f) := \sum_{j=1}^{m} f(j)$ as before, except that $\mathrm{ind}(\infty \ldots \infty) := -1$.

  e.g. $\mathrm{ind}(102\infty\hat{1}32\hat{2}\hat{3}1) = f(1) + f(2) + f(3) = 1 + 2 + 1 = 4$.

Let $\widetilde{\mathcal{C}}_{n,j}$ be the set of extended codes of length n with index j

Example

The extended codes of length 3.
$\widetilde{\mathcal{C}}_{3,-1} = \{\infty\infty\infty\}$, $\widetilde{\mathcal{C}}_{3,0} = \{0\infty\infty, \infty0\infty, \infty\infty0, \infty00, 0\infty0, 00\infty, 000\}$,
$\widetilde{\mathcal{C}}_{3,1} = \{1\hat{1}\infty, 1\infty\hat{1}, \infty1\hat{1}, 01\hat{1}, 10\hat{1}, 1\hat{1}0, 1\hat{1}1\}$, $\widetilde{\mathcal{C}}_{3,2} = \{11\hat{1}\}$.

An $\mathfrak{S}_n$-action as before makes $\widetilde{V}_{n,j} = \mathbb{C}\widetilde{\mathcal{C}}_{n,j}$ a permutation representation of $\mathfrak{S}_n$.

Theorem (L. 2022)

For $n \geq 1$, we have $\sum_{j=0}^{n} \mathrm{ch}(\widetilde{V}_{n,j-1}) t^j = \widetilde{Q}_n(\mathbf{x}, t)$.

Is there a permutation basis for $H^*(X_{\widetilde{\Sigma}_n})$ that has similar combinatorial structure as extended codes?

Extended codes

We define an *extended code* to be a code (α, f) s.t.

- α is over $\{0, 1, \ldots, m\} \cup \{\infty\}$ with ∞'s working as 0's in Stembridge codes,
- *index* $\mathrm{ind}(\alpha, f) := \sum_{j=1}^{m} f(j)$ as before, except that $\mathrm{ind}(\infty \ldots \infty) := -1$.

  e.g. $\mathrm{ind}(102\infty\hat{1}32\hat{2}\hat{3}1) = f(1) + f(2) + f(3) = 1 + 2 + 1 = 4$.

Let $\widetilde{\mathcal{C}}_{n,j}$ be the set of extended codes of length n with index j

Example

The extended codes of length 3.
$\widetilde{\mathcal{C}}_{3,-1} = \{\infty\infty\infty\}$, $\widetilde{\mathcal{C}}_{3,0} = \{0\infty\infty, \infty0\infty, \infty\infty0, \infty00, 0\infty0, 00\infty, 000\}$,
$\widetilde{\mathcal{C}}_{3,1} = \{1\hat{1}\infty, 1\infty\hat{1}, \infty1\hat{1}, 01\hat{1}, 10\hat{1}, 1\hat{1}0, 1\hat{1}1\}$, $\widetilde{\mathcal{C}}_{3,2} = \{11\hat{1}\}$.

An $\mathfrak{S}_n$-action as before makes $\widetilde{V}_{n,j} = \mathbb{C}\widetilde{\mathcal{C}}_{n,j}$ a permutation representation of $\mathfrak{S}_n$.

Theorem (L. 2022)

For $n \geq 1$, we have $\sum_{j=0}^{n} \mathrm{ch}(\widetilde{V}_{n,j-1}) t^j = \widetilde{Q}_n(\mathbf{x}, t)$.

Is there a permutation basis for $H^*(X_{\widetilde{\Sigma}_n})$ that has similar combinatorial structure as extended codes?

Basis for $H^*(X_{\widetilde{\Sigma}_n})$

Braden, Huh, Matherne, Proudfoot, Wang (2020) introduce the *augmented Chow ring* $\widetilde{A}(M)$ *of a matroid* M.

When M is the Boolean matroid B_n, the ring $\widetilde{A}(\mathsf{B}_n)$ is the same as $A(X_{\widetilde{\Sigma}_n})$. Hence by Danilov-Stembridge theorem,

$$\widetilde{A}(\mathsf{B}_n) \cong_{\mathfrak{S}_n} H^*(X_{\widetilde{\Sigma}_n}).$$

We find an analogue of the Feichtner-Yuzvinsky basis for $\widetilde{A}(M)$.

- The basis for $\widetilde{A}(\mathsf{B}_n)$ is

$$\widetilde{FY}(\mathsf{B}_n) = \left\{ x_{F_1}^{a_1} x_{F_2}^{a_2} \dots x_{F_k}^{a_k} : {}^{\emptyset = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \dots \subsetneq F_k \subseteq [n]}_{1 \le a_1 \le |F_1|,\ a_i \le |F_i| - |F_{i-1}| - 1 \text{ for } i \ge 2} \right\}.$$

e.g. $\widetilde{FY}(\mathsf{B}_3) =$
$\{1 \mid x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123} \mid x_1x_{123}, x_2x_{123}, x_3x_{123}, x_{12}^2, x_{13}^2, x_{23}^2, x_{123}^2 \mid x_{123}^3\}.$

The same $\mathfrak{S}_n$-action works on $\widetilde{FY}(\mathsf{B}_n)$.

Basis for $H^*(X_{\widetilde{\Sigma}_n})$

Braden, Huh, Matherne, Proudfoot, Wang (2020) introduce the *augmented Chow ring* $\widetilde{A}(M)$ *of a matroid* M.

When M is the Boolean matroid B_n, the ring $\widetilde{A}(\mathsf{B}_n)$ is the same as $A(X_{\widetilde{\Sigma}_n})$. Hence by Danilov-Stembridge theorem,

$$\widetilde{A}(\mathsf{B}_n) \cong_{\mathfrak{S}_n} H^*(X_{\widetilde{\Sigma}_n}).$$

We find an analogue of the Feichtner-Yuzvinsky basis for $\widetilde{A}(M)$.

- The basis for $\widetilde{A}(\mathsf{B}_n)$ is

$$\widetilde{FY}(\mathsf{B}_n) = \left\{ x_{F_1}^{a_1} x_{F_2}^{a_2} \cdots x_{F_k}^{a_k} : \substack{\emptyset = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \ldots \subsetneq F_k \subseteq [n] \\ 1 \le a_1 \le |F_1|,\ a_i \le |F_i| - |F_{i-1}| - 1 \text{ for } i \ge 2} \right\}.$$

e.g. $\widetilde{FY}(\mathsf{B}_3) =$
$\{1 \mid x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123} \mid x_1x_{123}, x_2x_{123}, x_3x_{123}, x_{12}^2, x_{13}^2, x_{23}^2, x_{123}^2 \mid x_{123}^3\}$.

The same $\mathfrak{S}_n$-action works on $\widetilde{FY}(\mathsf{B}_n)$.

The $\mathfrak{S}_n$-equivariant bijection

Let $\widetilde{\mathcal{C}}_n$ be the set of extended codes of length n. We construct a map $\widetilde{\phi} : \widetilde{FY}(\mathsf{B}_n) \to \widetilde{\mathcal{C}}_n$ that respects the $\mathfrak{S}_n$-action.

Theorem (L.)

The map $\widetilde{\phi} : \widetilde{FY}(\mathsf{B}_n) \to \widetilde{\mathcal{C}}_n$ is a bijection that respects the $\mathfrak{S}_n$-actions and takes the degree of the monomials to the index-1 of its image.

Example

Let $u_1 = x_{14}^{\mathbf{1}} x_{1247} x_{1245679}^{2} \in \widetilde{FY}(\mathsf{B}_9)$, then $\widetilde{\phi}(u_1)$ is

$$0\_\ \_0\_\ \_\ \_\ \_\ \_ \to 01\_0\_\ \_\hat{1}\_\ \_ \to 01\_022\hat{1}\_\hat{2} \to 01\infty022\hat{1}\infty\hat{2}.$$

Let $u_2 = x_{14}^{\mathbf{2}} x_{1247} x_{1245679}^{2} \in \widetilde{FY}(\mathsf{B}_9)$, then $\widetilde{\phi}(u_2)$ is

$$1\_\ \_\hat{1}\_\ \_\ \_\ \_\ \_ \to 12\_1\_\ \_\hat{2}\_\ \_ \to 12\_133\hat{2}\_\hat{3} \to 12\infty133\hat{2}\infty\hat{3}$$

- This also gives a bijective proof of the Shareshian–Wachs result that $\widetilde{Q}_n(\mathbf{x}, t)$ is the graded Frobenius series of $H^*(X_{\widetilde{\Sigma}_n})$.

Feichtner, Yuzvinsky's "package" related to stellohedron

In Feichtner and Yuzvnisky's theory of building set and Chow ring, the following comes like a package:

Package			
$\mathcal{L}$	building set $\mathcal{G}$	reduced nested set complex $\widetilde{\mathcal{N}}(\mathcal{L},\mathcal{G})$	$D(\mathcal{L},\mathcal{G})$
B_n	$B_n - \{\emptyset\}$	$\partial\Pi_n^*$	$A(X_{\Sigma_n})$
$\mathcal{L}(M)$	$\mathcal{L}(M) - \{\emptyset\}$	Bergman complex of M	$A(M)$
Braden, Huh, Matherne, Proudfoot, Wang, 2020			
		augmented Bergman complex of M	$\widetilde{A}(M)$
		aug. Berg. cpx of $\mathsf{B}_n \cong \partial\widetilde{\Pi_n}^*$	$\widetilde{A}(\mathsf{B}_n)$
Postnikov, Reiner, Williams, 2008			
B_{n+1}	graphical building set of n-star graph	$\widetilde{\mathcal{N}}(B_{n+1},\mathcal{B}(K_{1,n})) \cong \partial\widetilde{\Pi_n}^*$	

Augmented Chow ring of a matroid

Let M be a matroid on $[n]$ with lattice of flats $\mathcal{L}(M)$ and independence complex $\mathcal{I}(M)$.

The *augmented Chow ring of M* encodes information from both $\mathcal{L}(M)$ and $\mathcal{I}(M)$ and is defined as

$$\widetilde{A}(M) := \frac{\mathbb{Q}\left[\{x_F\}_{F\in\mathcal{L}(M)\setminus[n]} \cup \{y_1, y_2, \ldots, y_n\}\right]/(I_1 + I_2)}{\langle y_i - \sum_{F:i\notin F} x_F\rangle_{i=1,2,\ldots,n}} \quad (1)$$

where $I_1 = \langle x_F x_G : F, G$ are incomparable in $\mathcal{L}(M)\rangle$, $I_2 = \langle y_i x_F : i \notin F\rangle$.

- The numerator of (1) is the Stanley-Reisner ring of the augmented Bergman complex (fan) of M.

Augmented Bergman fan of a matroid

Definition: Let $I \in \mathcal{I}(M)$ and $\mathcal{F} = (F_1 \subsetneq \ldots \subsetneq F_k)$ be a chain in $\mathcal{L}(M)$.

- We say I is compatible with $\mathcal{F}$, denoted by $I \leq \mathcal{F}$, if $I \subseteq F_1$. In particular, $I \leq \emptyset$ for any $I \in \mathcal{I}(M)$.
- For $S \subseteq [n]$, write $e_S := \sum_{i \in S} e_i$.

The *augmented Bergman fan* $\widetilde{\Sigma}_M$ *of* M is a simplicial fan in $\mathbb{R}^n$ consisting of cones $\sigma_{I \leq \mathcal{F}}$ indexed by compatible pairs $I \leq \mathcal{F}$, where $\mathcal{F}$ is a chain in $\mathcal{L}(M) - \{[n]\}$ and

$$\sigma_{I \leq \mathcal{F}} = \mathbb{R}_{\geq 0}\left(\{e_i\}_{i \in I} \cup \{-e_{[n] \setminus F}\}_{F \in \mathcal{F}}\right).$$

The corresponding simplicial complex is called the *augmented Bergman complex*.

[Braden, Huh, Matherne, Proudfoot, Wang, 2020]: The augmented Bergman fan $\widetilde{\Sigma}_{B_n}$ is the normal fan of $\widetilde{\Pi}_n$.

Augmented Bergman fan of a matroid

Definition: Let $I \in \mathcal{I}(M)$ and $\mathcal{F} = (F_1 \subsetneq \ldots \subsetneq F_k)$ be a chain in $\mathcal{L}(M)$.

- We say I is compatible with $\mathcal{F}$, denoted by $I \leq \mathcal{F}$, if $I \subseteq F_1$. In particular, $I \leq \emptyset$ for any $I \in \mathcal{I}(M)$.
- For $S \subseteq [n]$, write $e_S := \sum_{i \in S} e_i$.

The *augmented Bergman fan $\widetilde{\Sigma}_M$ of M* is a simplicial fan in $\mathbb{R}^n$ consisting of cones $\sigma_{I \leq \mathcal{F}}$ indexed by compatible pairs $I \leq \mathcal{F}$, where $\mathcal{F}$ is a chain in $\mathcal{L}(M) - \{[n]\}$ and

$$\sigma_{I \leq \mathcal{F}} = \mathbb{R}_{\geq 0}\left(\{e_i\}_{i \in I} \cup \{-e_{[n] \setminus F}\}_{F \in \mathcal{F}}\right).$$

The corresponding simplicial complex is called the *augmented Bergman complex*.

[Braden, Huh, Matherne, Proudfoot, Wang, 2020]: The augmented Bergman fan $\widetilde{\Sigma}_{\mathrm{B}_n}$ is the normal fan of $\widetilde{\Pi}_n$.

Augmented Bergman fan of a matroid

Definition: Let $I \in \mathcal{I}(M)$ and $\mathcal{F} = (F_1 \subsetneq \ldots \subsetneq F_k)$ be a chain in $\mathcal{L}(M)$.

- We say I is compatible with $\mathcal{F}$, denoted by $I \leq \mathcal{F}$, if $I \subseteq F_1$. In particular, $I \leq \emptyset$ for any $I \in \mathcal{I}(M)$.
- For $S \subseteq [n]$, write $e_S := \sum_{i \in S} e_i$.

The *augmented Bergman fan $\widetilde{\Sigma}_M$ of M* is a simplicial fan in $\mathbb{R}^n$ consisting of cones $\sigma_{I \leq \mathcal{F}}$ indexed by compatible pairs $I \leq \mathcal{F}$, where $\mathcal{F}$ is a chain in $\mathcal{L}(M) - \{[n]\}$ and

$$\sigma_{I \leq \mathcal{F}} = \mathbb{R}_{\geq 0} \left(\{e_i\}_{i \in I} \cup \{-e_{[n] \setminus F}\}_{F \in \mathcal{F}} \right).$$

The corresponding simplicial complex is called the *augmented Bergman complex*.

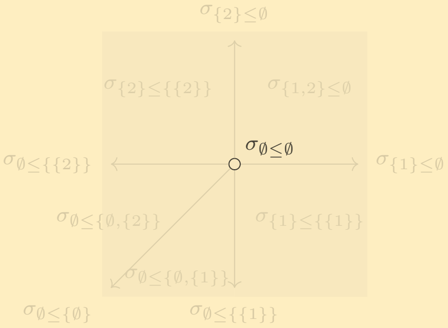
[Braden, Huh, Matherne, Proudfoot, Wang, 2020]: The augmented Bergman fan $\widetilde{\Sigma}_{\mathsf{B}_n}$ is the normal fan of $\widetilde{\Pi}_n$.

Stellohedron as a dual augmented Bergman complex

$$\sigma_{I\leq\mathcal{F}} \coloneqq \mathbb{R}_{\geq 0}\left(\{e_i\}_{i\in I}\cup\{-e_{[n]\setminus F}\}_{F\in\mathcal{F}}\right)$$

Example

Boolean matroid B_2, $\mathcal{I}(\mathrm{B}_2)=\{\emptyset,1,2,12\}$. The augmented Bergman complex is $\partial\widetilde{\Pi}_2^*$.



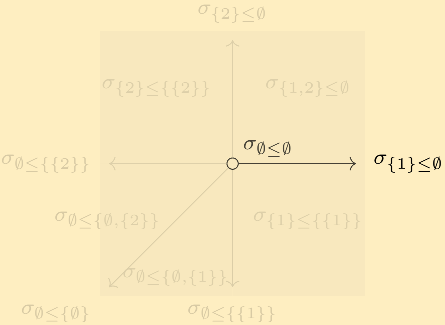
.

Stellohedron as a dual augmented Bergman complex

$$\sigma_{I\le\mathcal{F}} \coloneqq \mathbb{R}_{\ge 0}\left(\{e_i\}_{i\in I} \cup \{-e_{[n]\setminus F}\}_{F\in\mathcal{F}}\right)$$

Example

Boolean matroid B_2, $\mathcal{I}(\mathsf{B}_2) = \{\emptyset, 1, 2, 12\}$. The augmented Bergman complex is $\partial\widetilde{\Pi}_2^*$.



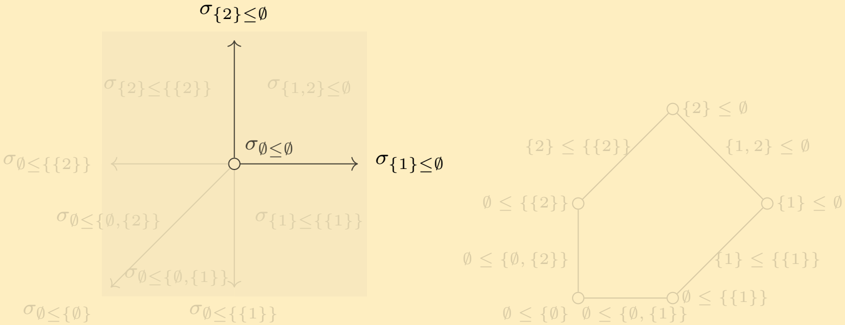
.

Stellohedron as a dual augmented Bergman complex

$\sigma_{I\leq\mathcal{F}} \coloneqq \mathbb{R}_{\geq 0}\left(\{e_i\}_{i\in I} \cup \{-e_{[n]\setminus F}\}_{F\in\mathcal{F}}\right)$

Example

Boolean matroid B_2, $\mathcal{I}(\mathsf{B}_2) = \{\emptyset, 1, 2, 12\}$. The augmented Bergman complex is $\partial\widetilde{\Pi}_2^*$.



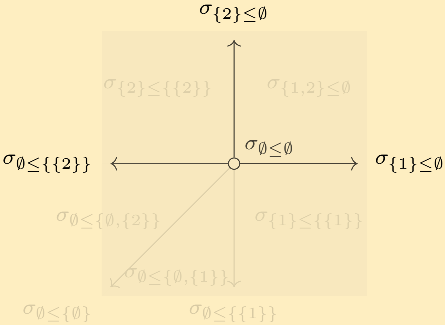
.

Stellohedron as a dual augmented Bergman complex

$\sigma_{I \leq \mathcal{F}} \coloneqq \mathbb{R}_{\geq 0}\left(\{e_i\}_{i \in I} \cup \{-e_{[n] \setminus F}\}_{F \in \mathcal{F}}\right)$

Example

Boolean matroid B_2, $\mathcal{I}(\mathsf{B}_2) = \{\emptyset, 1, 2, 12\}$. The augmented Bergman complex is $\partial\widetilde{\Pi}_2^*$.



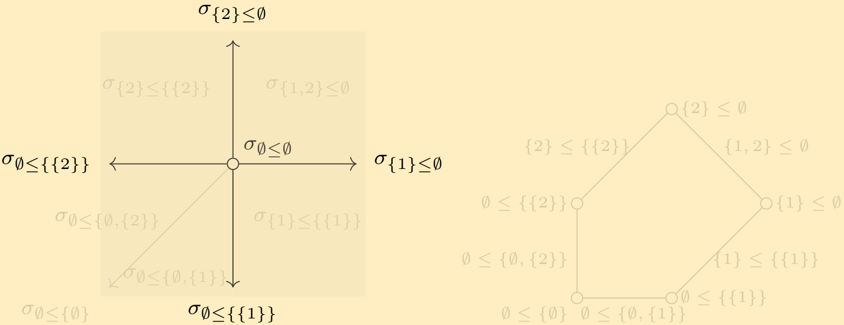
.

Stellohedron as a dual augmented Bergman complex

$$\sigma_{I\le\mathcal{F}} := \mathbb{R}_{\ge 0}\left(\{e_i\}_{i\in I}\cup\{-e_{[n]\setminus F}\}_{F\in\mathcal{F}}\right)$$

Example

Boolean matroid B_2, $\mathcal{I}(\mathsf{B}_2)=\{\emptyset,1,2,12\}$. The augmented Bergman complex is $\partial\widetilde{\Pi}_2^*$.



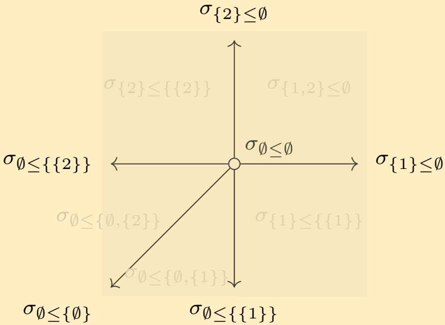
.

Stellohedron as a dual augmented Bergman complex

$$\sigma_{I\le\mathcal{F}} \coloneqq \mathbb{R}_{\ge 0}\left(\{e_i\}_{i\in I}\cup\{-e_{[n]\setminus F}\}_{F\in\mathcal{F}}\right)$$

Example

Boolean matroid B_2, $\mathcal{I}(\mathsf{B}_2)=\{\emptyset,1,2,12\}$. The augmented Bergman complex is $\partial\widetilde{\Pi}_2^*$.



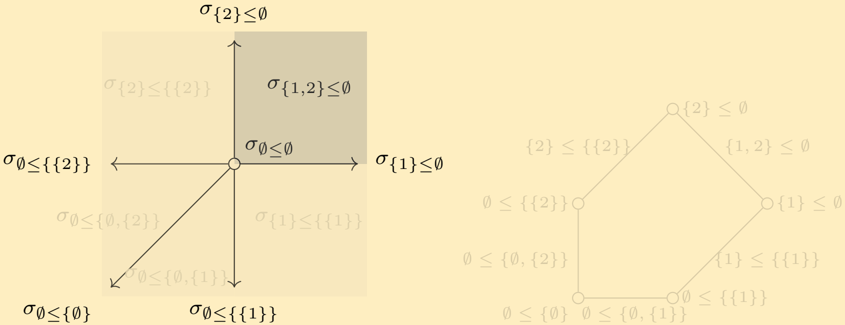
.

Stellohedron as a dual augmented Bergman complex

$\sigma_{I \le \mathcal{F}} := \mathbb{R}_{\ge 0}\left(\{e_i\}_{i \in I} \cup \{-e_{[n]\setminus F}\}_{F \in \mathcal{F}}\right)$

Example

Boolean matroid B_2, $\mathcal{I}(B_2) = \{\emptyset, 1, 2, 12\}$. The augmented Bergman complex is $\partial\widetilde{\Pi}_2^*$.



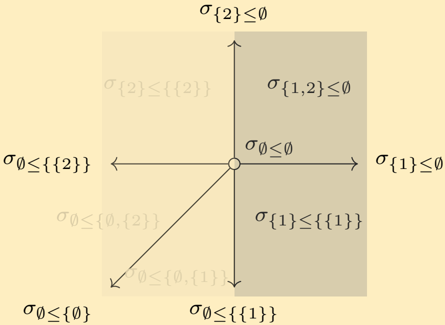
.

Stellohedron as a dual augmented Bergman complex

$$\sigma_{I \leq \mathcal{F}} \coloneqq \mathbb{R}_{\geq 0}\left(\{e_i\}_{i \in I} \cup \{-e_{[n]\setminus F}\}_{F \in \mathcal{F}}\right)$$

Example

Boolean matroid B_2, $\mathcal{I}(\mathsf{B}_2) = \{\emptyset, 1, 2, 12\}$. The augmented Bergman complex is $\partial\widetilde{\Pi}_2^*$.



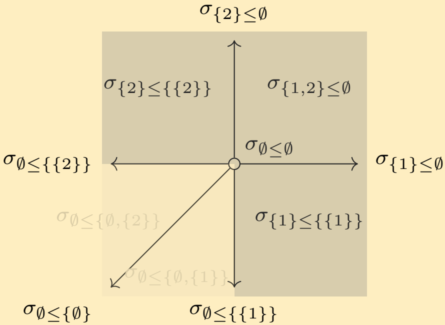
.

Stellohedron as a dual augmented Bergman complex

$\sigma_{I\leq\mathcal{F}} \coloneqq \mathbb{R}_{\geq 0}\left(\{e_i\}_{i\in I}\cup\{-e_{[n]\setminus F}\}_{F\in\mathcal{F}}\right)$

Example

Boolean matroid B_2, $\mathcal{I}(\mathrm{B}_2)=\{\emptyset,1,2,12\}$. The augmented Bergman complex is $\partial\widetilde{\Pi}_2^*$.



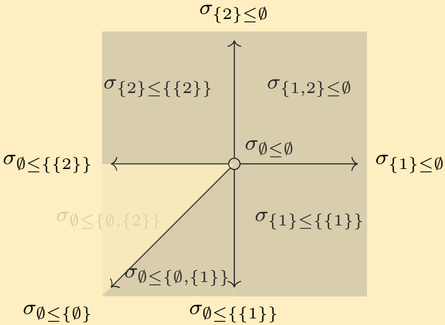
.

Stellohedron as a dual augmented Bergman complex

$\sigma_{I \leq \mathcal{F}} \coloneqq \mathbb{R}_{\geq 0} \left(\{e_i\}_{i \in I} \cup \{-e_{[n] \setminus F}\}_{F \in \mathcal{F}} \right)$

Example

Boolean matroid B_2, $\mathcal{I}(\mathrm{B}_2) = \{\emptyset, 1, 2, 12\}$. The augmented Bergman complex is $\partial \widetilde{\Pi}_2^*$.



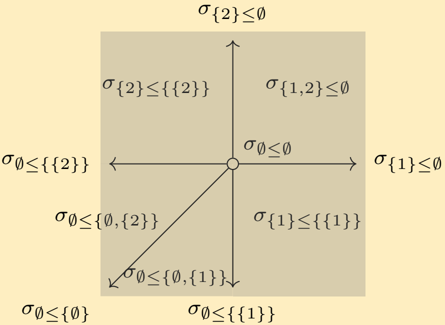
.

Stellohedron as a dual augmented Bergman complex

$$\sigma_{I \le \mathcal{F}} := \mathbb{R}_{\ge 0}\left(\{e_i\}_{i \in I} \cup \{-e_{[n]\setminus F}\}_{F \in \mathcal{F}}\right)$$

Example

Boolean matroid B_2, $\mathcal{I}(\mathrm{B}_2) = \{\emptyset, 1, 2, 12\}$. The augmented Bergman complex is $\partial \widetilde{\Pi}_2^*$.



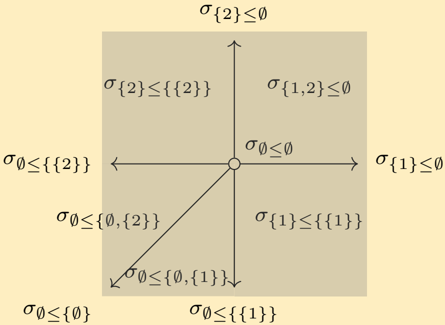
.

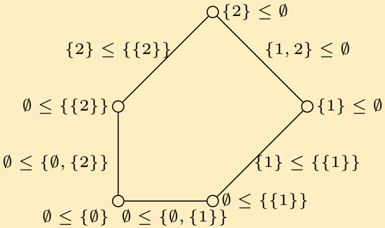
Stellohedron as a dual augmented Bergman complex

$\sigma_{I \leq \mathcal{F}} \coloneqq \mathbb{R}_{\geq 0}\left(\{e_i\}_{i \in I} \cup \{-e_{[n]\setminus F}\}_{F \in \mathcal{F}}\right)$

Example

Boolean matroid B_2, $\mathcal{I}(\mathsf{B}_2) = \{\emptyset, 1, 2, 12\}$. The augmented Bergman complex is $\partial\widetilde{\Pi}_2^*$.





.

Stellohedron $\widetilde{\Pi}_n$ as a dual nested set complex

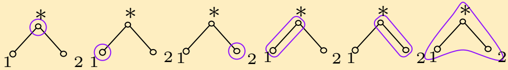
[Postnikov, Reiner, Williams, 2008]:
The *n-star graph* $K_{1,n} := (V, E)$ with $V = [n] \cup \{*\}$ and $E = \{\{i, *\} : i \in [n]\}$.
Consider the *graphical building set*

$$\mathcal{B}(K_{1,n}) := \{I \subset V : \text{the induced subgraph on } I \text{ is connected}\},$$

then the reduced nested set complex $\widetilde{\mathcal{N}}(B_{n+1}, \mathcal{B}(K_{1,n}))$ is combinatorially equivalent to $\partial\widetilde{\Pi}_n^*$.

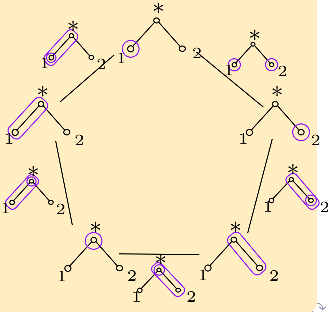
Example

$\mathcal{B}(K_{1,2})$ consists of the following elements:



.

The corresponding reduced nested set complex is $\partial\widetilde{\Pi_2}^*$.



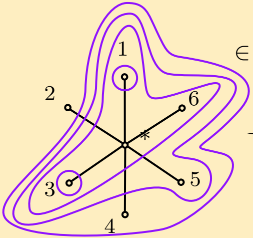
Connection between $\widetilde{\Sigma}_{\mathsf{B}_n}$ and $\widetilde{\mathcal{N}}(B_{n+1}, \mathcal{B}(K_{1,n}))$

Proposition (L. 2022)

There is a poset isomorphism between the face lattice of the augmented Bergman fan $\widetilde{\Sigma}_{\mathsf{B}_n}$ and that of the reduced nested set complex $\widetilde{\mathcal{N}}(B_{n+1}, \mathcal{B}(K_{1,n}))$.

Example

$$\widetilde{\mathcal{N}}\left(\mathsf{B}_{n+1}, B_{K_{1,n}}\right) \longleftrightarrow \left\{\sigma_{I\leq\mathcal{F}} \;:\; \begin{matrix} I\in\mathcal{I}(\mathsf{B}_n), \\ \mathcal{F} \text{ is a flag of proper subsets of } [n], \\ I\subset\min(\mathcal{F}) \end{matrix}\right\}$$



$$\in \widetilde{\mathcal{N}}\left(B_7, \mathcal{B}_{K_{1,6}}\right)$$

$$\longleftrightarrow \{1,3\} \leq \{\{1,3,6\} \subset \{1,3,5,6\} \subset \{1,3,4,5,6\}\}$$

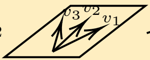
Connection for general matroids

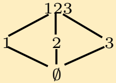
Let M be a matroid with lattice of flats $\mathcal{L}(M)$ and independence complex $\mathcal{I}(M)$.

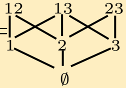
We construct a new poset $\widetilde{\mathcal{L}}(M)$ from $\mathcal{L}(M)$ and $\mathcal{I}(M)$:

- As a set, $\widetilde{\mathcal{L}}(M) = \mathcal{L}(M) \uplus \mathcal{I}(M)$. Write $F \in \mathcal{L}(M)$ as F_* in $\widetilde{\mathcal{L}}(M)$.
- For $I \in \mathcal{I}(M)$, define $I \lessdot \mathrm{cl}_M(I)_*$ where $\mathrm{cl}_M(I)$ is the closure of I in M. The relations inside $\mathcal{L}(M), \mathcal{I}(M)$ stay the same.

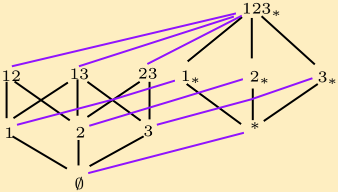
Example

$M = U_{3,2}$ 

$\rightsquigarrow \mathcal{L}(U_{3,2}) =$ 

$, \mathcal{I}(U_{3,2}) =$ 

Then the new poset is

$\widetilde{\mathcal{L}}(U_{3,2}) =$ 

Connection for general matroids

Take $\widetilde{\mathcal{G}} = \{\{1\},\ldots,\{n\}\} \cup \{F_*\}_{F\in\mathcal{L}(M)}$ as the building set in $\widetilde{\mathcal{L}}(M)$, then all faces of the reduced nested set complex are of the form

$$\{\{i\}\}_{i\in I} \cup \{F_*\}_{F\in\mathcal{F}}$$

for some compatible pair $I \le \mathcal{F}$ where $I \in \mathcal{I}(M)$ and $\mathcal{F}$ is a chain of $\mathcal{L}(M)$.

Theorem (L. ; Eur, Huh, Larson 2022)

1. *There is a poset isomorphism between the face lattices of $\widetilde{\mathcal{N}}(\widetilde{\mathcal{L}}(M),\widetilde{\mathcal{G}})$ and $\widetilde{\Sigma}_M$:*

   $$\{\{i\}\}_{i\in I} \cup \{F_*\}_{F\in\mathcal{F}} \longleftrightarrow \sigma_{I\le\mathcal{F}}$$

   for compatible pair $I \le \mathcal{F}$ where $I \in \mathcal{I}(M)$ and chain $\mathcal{F} \subset \mathcal{L}(M) - \{[n]\}$ of proper flats.
2. $D(\widetilde{L}(M),\widetilde{\mathcal{G}}) = \widetilde{A}(M)$.

This connection was also independently found by Chris Eur and later included in his recent preprint with Huh and Larson

Analogue of the Feichtner-Yuzvinsky basis

We apply Feichtner-Yuzvinsky's basis to the chow ring $D(\widetilde{\mathcal{L}}(M), \widetilde{\mathcal{G}})$ and obtain:

Corollary (L. 2022; Eur, Huh, Larson, 2022)

The augmented Chow ring $\widetilde{A}(M)$ of M has the following basis

$$\widetilde{FY}(M) := \left\{ x_{F_1}^{a_1} x_{F_2}^{a_2} \ldots x_{F_k}^{a_k} : {}_{1 \le a_1 \le \mathrm{rk}(F_1),\ a_i \le \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1 \text{ for } i \ge 2}^{\emptyset = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \ldots \subsetneq F_k} \right\}$$

Summary

$\mathcal{L}$	building set $\mathcal{G}$	reduced nested set complex $\widetilde{\mathcal{N}}(\mathcal{L},\mathcal{G})$	$D(\mathcal{L},\mathcal{G})$
B_n	$B_n - \{\emptyset\}$	$\partial\Pi_n^*$	$A(X_{\Sigma_n})$
$\mathcal{L}(M)$	$\mathcal{L}(M) - \{\emptyset\}$	Bergman complex of M	$A(M)$
		aug. Berg. cpx of $\mathrm{B}_n \cong \partial\widetilde{\Pi_n}^*$	$\widetilde{A}(\mathrm{B}_n)$
B_{n+1}	graphical building set of n-star graph	$\widetilde{\mathcal{N}}(B_{n+1},\mathcal{B}(K_{1,n})) \cong \partial\widetilde{\Pi_n}^*$	
		augmented Bergman complex of M	$\widetilde{A}(M)$
$\widetilde{\mathcal{L}}(M)$	$\widetilde{\mathcal{G}}$	$\widetilde{\mathcal{N}}(\widetilde{\mathcal{L}}(M),\widetilde{\mathcal{G}})$	

Consequently, $\widetilde{A}(M) = D(\widetilde{\mathcal{L}}(M),\widetilde{\mathcal{G}})$ and hence has an FY-basis.

The End

Thank you!

Related works

- Since $X_{\Sigma_n} \cong$ regular semisimple $\mathrm{Hess}(S, h)$ with $h = (2, 3, \ldots, n, n)$, answering Stembridge's question gives a "dream" solution to a special case of the Stanley-Stembridge conjecture.
- Erasing Marks Conjecture : Chow (2015), using GKM theory, conjectured that some classes in $H_T^*(X_{\Sigma_n})$ when descending to $H^*(X_{\Sigma_n})$ give such a basis.
- Cho, Hong, and Lee (2020) proved the conjecture. It will be interesting to see the relationship between our basis and theirs.