Hilbert–Poincare series of matroid Chow rings and intersection cohomology

Lorenzo Vecchi
lorenzo.vecchi6@unibo.it

Alma Mater Studiorum - Università di Bologna

July 19th, 2023

Plan

1. Find new polynomial invariants of matroids by taking Hilbert series
2. Study their combinatorial properties

This is a joint work with Luis Ferroni (KTH), Jacob Matherne (U of Bonn, Max Planck Institute), and Matthew Stevens (U of Bonn)

Geometric lattices

A *geometric lattice* is a ranked lattice which is

- atomistic: $x = \bigvee_{a \leq x} a$
- semimodular: $\mathrm{rk}(x) + \mathrm{rk}(y) \geq \mathrm{rk}(x \vee y) + \mathrm{rk}(x \wedge y)$

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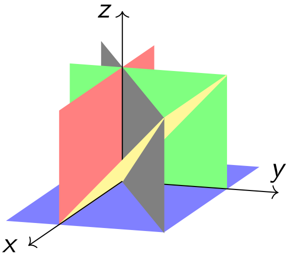
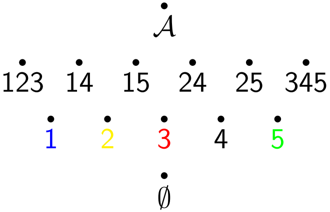
Example

Hyperplane arrangements

The poset of intersections of an arrangement $\mathcal{A}$ is a geometric lattice



$\mathcal{L}(\mathcal{A}) =$



Matroids as posets

There is a correspondence between the class of geometric lattices and the class of matroids.

$$\mathrm{M} \rightsquigarrow \mathcal{L}(\mathrm{M})$$

$\mathcal{L}(\mathrm{M})$ is called the *lattice of flats of* M.

Characteristic polynomial

On the lattice $\mathcal{L}(\mathsf{M})$, we can compute the *characteristic polynomial*

$$\chi_{\mathsf{M}}(x) = x^3 - 5x^2 + 8x - 4$$

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Theorem (Orlik-Solomon)

If $\mathcal{A}$ is a complex arrangement, then $\mathcal{M}(\mathcal{A}) = \mathbb{C}^d \setminus \bigcup_{H \in \mathcal{A}} H$ is a smooth variety and

$$(-x)^{\mathrm{rk}\,\mathrm{M}} \chi_M(-x^{-1}) = \mathrm{Hilb}(\mathrm{H}^*(\mathcal{M}(\mathcal{A})))$$

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Theorem (Heron-Rota-Welsh, Adiprasito-Huh-Katz'18)

$\chi_{\mathrm{M}}(x)$ *is log-concave for every* M, *i.e.* $w_j^2 \geq w_{j-1} w_{j+1}$.

Find new polynomials by taking Hilbert series associated to matroids.

The Chow ring

Definition (Feichtner-Yuzvinsky'04)

The Chow ring of M is defined as the quotient ring

$$\underline{\mathrm{CH}}(\mathrm{M}) := \mathbb{Q}[x_F \mid F \in \mathcal{L}(\mathrm{M}) \setminus \{\emptyset, E\}] \Big/ (I + J),$$

$$I = \langle x_{F_1} x_{F_2} \mid F_1, F_2 \text{ are incomparable} \rangle \,,$$

$$J = \left\langle \sum_{F \ni i} x_F - \sum_{F \ni j} x_F \mid i, j \in E \right\rangle .$$

The augmented Chow ring

Definition (Braden-Huh-Matherne-Proudfoot-Wang'22)

The *augmented* Chow ring of M is defined as the quotient ring

$$\mathrm{CH}(\mathrm{M}) = \mathbb{Q}[x_F,\, y_i | F \in \mathcal{L}(\mathrm{M}) \setminus \{E\} \text{ and } i \in E] \Big/ (I + J + K),$$

$$I = \left\langle y_i - \sum_{F \not\ni i} x_F \,\middle|\, i \in E \right\rangle,$$

$$J = \langle x_{F_1} x_{F_2} \,|\, F_1, F_2 \text{ are incomparable} \rangle,$$

$$K = \langle y_i x_F | \, i \notin F \rangle.$$

What makes this interesting?

Theorem (AHK'18, BHMPW'22)

(PD), (HL) and (HR) hold for every $\underline{\mathrm{CH}}(\mathrm{M})$ *and* $\mathrm{CH}(\mathrm{M})$.

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(HR) $\Rightarrow \chi_{\mathsf{M}}(x)$ *is log-concave.*

$$\underline{\mathrm{H}}_{\mathsf{M}}(x) = \sum_{j=0}^{\operatorname{rk}\mathsf{M}-1} \dim_{\mathbb{Q}}(\underline{\mathrm{CH}}^j(\mathsf{M}))x^j \qquad \mathrm{H}_{\mathsf{M}}(x) = \sum_{j=0}^{\operatorname{rk}\mathsf{M}} \dim_{\mathbb{Q}}(\mathrm{CH}^j(\mathsf{M}))x^j.$$

$\underline{\mathrm{H}}_{\mathsf{M}}(x)$ and $\mathrm{H}_{\mathsf{M}}(x)$ are called respectively the *Chow polynomial* and *augmented Chow polynomial*.

Some combinatorial properties

- (PD) $\rightsquigarrow$ $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ and $\mathrm{H}_{\mathsf{M}}(x)$ are palindromic.
- (HL) $\rightsquigarrow$ $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ and $\mathrm{H}_{\mathsf{M}}(x)$ are unimodal.

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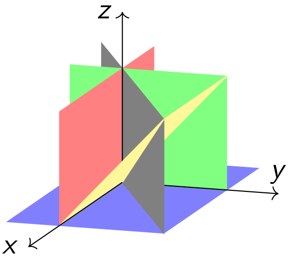

$$\underline{\mathrm{H}}_{\mathsf{M}}(x) = x^2 + 7x + 1$$
$$\mathrm{H}_{\mathsf{M}}(x) = x^3 + 12x^2 + 12x + 1$$

Some examples I

If M is a Boolean matroid on n elements then

$$\underline{\mathrm{H}}_{\mathsf{M}}(x) = A_n(x),$$
$$\mathrm{H}_{\mathsf{M}}(x) = \widetilde{A}_n(x).$$

These are the *Eulerian* and *binomial Eulerian* polynomials.

$$A_n(x) = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{exc}(\sigma)} \qquad \widetilde{A}_n(x) = 1 + x \sum_{j=1}^{n} \binom{n}{j} A_j(x).$$

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Theorem (Frobenius, Haglund-Zhang'20, Brändén-Jochemko'22)

The polynomials $A_n(x)$ and $\widetilde{A}_n(x)$ are always real-rooted.

Some examples II

If M is a uniform matroid on n elements and rank $n-1$:

$$\underline{\mathrm{H}}_{\mathsf{M}}(x) = \frac{1}{x} d_n(x),$$
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These are the *derangement* and *Eulerian* polynomials.

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Theorem (Brenti'92, Canfield'94, Zhang'94)

The polynomial $d_n(x)$ is always real-rooted.

Conjecture (Huh'20, Stevens'21, Ferroni-Schröter'22)

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The Kazhdan–Lusztig polynomial $P_{\mathsf{M}}(x)$ *and* $Z_{\mathsf{M}}(x)$*, the Hilbert series of the intersection cohomology module* $\mathrm{IH}(\mathsf{M})$ *are real-rooted.*

The real deal with matroids

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Problems

- We have no fast, closed formula to compute them
- It is computationally expensive to build the Chow rings.

Goal: Compute them on $\mathcal{L}(M)$ efficiently without actually passing through the Chow rings!

Combinatorial formulas

Theorem (FMSV'23)

Let M *be a loopless matroid, then* $\underline{\mathrm{H}}_{\mathrm{M}}(x)$ *and* $\mathrm{H}_{\mathrm{M}}(x)$ *satisfy the following*

- *If* $\mathrm{rk}(\mathrm{M}) = 0$*, then* $\underline{\mathrm{H}}_{\mathrm{M}}(x) = 1$.
- *If* $\mathrm{rk}(\mathrm{M}) > 0$*, then* $\deg \underline{\mathrm{H}}_{\mathrm{M}}(x) = \mathrm{rk}(\mathrm{M}) - 1$ *and*

$$\underline{\mathrm{H}}_{\mathrm{M}}(x) = \sum_{\substack{F \in \mathcal{L}(\mathrm{M}) \\ F \neq \emptyset}} \frac{\chi_{\mathrm{M}|_F}(x)}{x-1} \underline{\mathrm{H}}_{\mathrm{M}/F}(x) \qquad \mathrm{H}_{\mathrm{M}}(x) = \sum_{F \in \mathcal{L}(\mathrm{M})} x^{\mathrm{rk}(F)} \underline{\mathrm{H}}_{\mathrm{M}/F}(x).$$

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- In the incidence algebra: $\delta = \overline{\chi} * \underline{\mathrm{H}} \qquad \mathrm{H} = \widetilde{\zeta} * \underline{\mathrm{H}}$
- We actually provide a setting in which $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ and $\mathrm{H}_{\mathsf{M}}(x)$ are "non-singular" analogous of $P_{\mathsf{M}}(x)$ and $Z_{\mathsf{M}}(x)$.

Some examples III

If M is a uniform matroid on n elements and rank k

Theorem (FMSV'23)

$$\underline{\mathrm{H}}_{\mathrm{U}_{k,n}}(x) = \sum_{j=0}^{k-1} \binom{n}{j} d_j(x)\,(1 + x + \cdots + x^{k-1-j})$$

$$\mathrm{H}_{\mathrm{U}_{k,n}}(x) = 1 + x \sum_{j=0}^{k-1} \binom{n}{j} A_j(x)\,(1 + x + \cdots + x^{k-1-j}).$$

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Theorem (FMSV'23)

The polynomial $\mathrm{H}_{\mathsf{U}_{k,n}}(x)$ *is always real-rooted.*

Proof requires using a general result of Haglund and Zhang.

Some more evidence

Theorem (FMSV'23)

If M *is a sparse paving matroid with at most* 40 *elements, then* $\underline{\mathrm{H}}_{\mathrm{M}}(x)$ *and* $\mathrm{H}_{\mathrm{M}}(x)$ *are real-rooted.*

Conjecture

100% of the matroids are sparse paving.

γ-polynomial

If f is palindromic of degree d

$$f(x) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i x^i (x+1)^{d-2i}$$

$$\gamma(f,x) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i x^i.$$

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Theorem

If $f \in \mathbb{Z}_{\geq 0}[x]$ is palindromic,
f is real-rooted $\Rightarrow$ f is γ-positive $\Rightarrow$ f is unimodal.

γ-positivity

Theorem (FMSV'23)

$\underline{\mathrm{H}}_{\mathsf{M}}(x)$ *and* $\mathrm{H}_{\mathsf{M}}(x)$ *are* γ*-positive for every* M.

γ-positivity

Theorem (FMSV'23)

$\underline{\mathrm{H}}_{\mathsf{M}}(x)$ and $\mathrm{H}_{\mathsf{M}}(x)$ are γ-positive for every M*.*

- Proof: induction linked to a semi-small decompositions of $\underline{\mathrm{CH}}(\mathsf{M})$ and $\mathrm{CH}(\mathsf{M})$ (BHMPW'22). Also observed by Wang.
- Gives self-contained independent proof of the γ-positivity of derangement, Eulerian and binomial Eulerian polynomials.
- With a different induction we also show γ-positivity for the intersection cohomology module $\mathrm{IH}(\mathsf{M})$.

Thank you! ☺